A robust rank test for location under asymmetry

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Abstract

We propose a winsorized adaptative signed rank test for the location alternative for samples coming from asymmetric distributions. We give conditions under which the proposed test can have either greater or smaller acceptance breakdown, than the acceptance breakdown of a recently appeared adaptative rank test for location under asymmetry. Moreover, when symmetry of the sampled distribution can be justified the proposed test has greater acceptance breakdown than the winsorized signed rank test.

***Key words*:** Acceptance breakdown, Location tests, Winsorized rank tests..

1. Introducción

Let $X_1, \ldots, X_N$, a random sample from a continuous distribution $F(x-\theta)$ such that $F(0) = 1/2$ uniquely. Without loss of generality consider the test problem:

$$H_0 : \theta = 0 \quad vs. \quad H_1 : \theta > 0. \tag{1}$$

Under such general conditions about F, the Sign Test is commonly used to test H_0. When symmetry of F around zero is justifiable the Wilcoxon Signed Rank Test is preferred because it is more powerful than the Sign Test (see for example Randles & Wolfe (1979)). Moreover, if the sampled distribution is some symmetrical distribution the Winsorizad Signed Rank Test is preferred, with small winsorization parameter when it is close to the normal distribution, and with larger winsorization parameter when the sampled distribution is closer to the double exponential (see Hettmansperger (1984) pág 92-93). Baklizi (2005) proposes an adaptative robust rank test for location when the sampled distribution is asymmetric and shows by a simulation study, that his test is more powerful than other competitors for the same purpose in most cases he has considered. There is also in the same paper an

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interesting discussion about the problems of some other adaptative tests for the same problem.

We mix the Baklizit's modification of the Wilcoxon scores with the Tukeyt's winsorization technique to produce a new winsorized rank test, which can have greater or less acceptance breakdown than the Baklizit's Test depending on a symmetry parameter. Moreover, when the symmetry of the sampled distribution is justifiable, our test is more tolerant than the well known Winsorized Signed Rank Test.

2. Proposed test statistic and its properties.

Let $|X|_{(1)} \leq \cdots \leq |X|_{(N)}$ be the sequence of ordered absolute values of the sample, define R_i the rank of $|X_i|$ by $|X_i| = |X|_{(R_i)}$ and the antirank D_j of $|X|_{(j)}$ by $|X_{D_j}| = |X|_{(j)}$. Let $s(X_i)$ be the indicator variables:

$$s(X_j) = \begin{cases} 1 & \text{if } X_{D_j} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Scores statistics are defined for a nonconstant sequence $0 = a(0) \leq a(1) \leq \cdots \leq a(N)$ by $\sum_{i=1}^{N} a(R_i)s(X_i)$. Scores can be given by a score generating function taking $a(i) = \phi(i/(N+1))$ and then it can be defined a statistic generated by the score generating function as (see Hettmansperger (1984)):

$$\bar{V} = \frac{1}{N} \sum_{i=1}^{N} \phi\left(\frac{R_i}{N+1}\right) s(X_i)$$

where $\phi(u)$, $0 < u < 1$ is a nonnegative and nondecreasing function such that $\int_0^1 \phi(u)\, du < 1$ and $0 < \int_0^1 \phi^2(u)\, du < \infty$.

Some known special cases are: the Sign Test (S-Test) $\phi(u) = 1$, the Wilcoxon Signed Rank Test (W-Test) $\phi(u) = u$, the Winsorized Signed Rank Test (WW-Test) $\phi(u) = \min\{u, 1-\gamma\}$.

Another special two cases of $\bar{V}$ are the Baklizi Test (B-Test) $\phi(u) = u^p$ and the Proposed Test (CV-test) $\phi(u) = \min\{u^{g(p)}, 1-\gamma\}$, where p is the p-value of a test for the hypothesis of symmetry $F(x) = 1 - F(-x)$ for all x against asymmetric alternatives, proposed in Randles et al. (1980).

Using Baklizi (2005) notation, let P be the random variable denoting the p-value of the Randles test for symmetry. For a fixed value of $P = p$, Baklizi uses the following conditional (on p) score function:

$$a^*(R_i) = R_i^p. \tag{2}$$

where p is used as indicator of the asymmetry of F. Note that the shorter is the p-value, the greater is the asymmetry of F.

The proposed test statistic uses the following winsorized score function:

$$a(R_i) = \min\left\{\left(\frac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} \tag{3}$$

where $0 < g(p) \leq 1$ and γ is the proportion of observations whose ranks will be replaced by $1-\gamma$.

Both conditional test statistics can be written as follows:

Baklizi statistic

$$\bar{B} = \frac{1}{N(N+1)^p}\sum_{i=1}^{N} {R_i}^p s(X_i).$$

Proposed statistic

$$\overline{CV} = \frac{1}{N}\sum_{i=1}^{N} \min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} s(X_i). \tag{4}$$

The proposed test rejects H_0 in favor of H_1 for a given α, when $\overline{CV} \geq k$, where k is determined such that $P(\overline{CV} \geq k \mid p) = \alpha$. The overall size of the test is α because for a fixed γ (see Baklizi (2005)):

$$P(\overline{CV} \geq k) = \int_0^1 P(\overline{CV} \geq k \mid p) f(p) dp = \alpha$$

The exact conditional distribution of the test statistic under H_0 can be obtained by enumeration as follows: For fixed p and γ, let $Z = \{0,1\}^N$ be the set of all posible configurations of 1s and 0s assignable to the sample values arranged in a $2^N \times N$ matrix, such that each row corresponds to a different configuration, and for $a(i)$ as in (3) let $R^{'} = (a(1), \ldots, a(N))$ the vector of scores. Let z any row of Z then the values of $\overline{CV}$ can be obtained as the product $\overline{CV}(z) = zR$, and so the critical values for the test can be calculated from:

$$P(\overline{CV}(z) \leq m \mid p) = \frac{\sharp(z \in Z : \overline{CV}(z) \leq m)}{2^N}$$

The proposed test statistic has the following properties:

a) $\phi(u) = \min\{u^{g(p)}, 1-\gamma\}$ is a well defined score function and satisfies (Proof in Appendix A.1.)

$$\int_0^1 \phi(u)\, du < 1-\gamma,$$

and

$$0 < \int_0^1 \phi^2(u)\, du < 1$$

b) From the theory for linear rank statistics the conditional mean and variance exact and asymptotic of $\overline{CV}$ for a given p under H_0 are (see Hettmansperger (1984), pag. 88):

$$E\left[\,\overline{CV} \mid p\,\right] = \frac{1}{2N}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} \tag{5}$$

$$\xrightarrow{N\to\infty} \quad \frac{(1-\gamma)}{2}\left[1-\frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] \tag{6}$$

$$N\,Var\left[\,\overline{CV} \mid p\,\right] = \frac{1}{4N}\sum_{i=1}^{N}\min^2\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} \tag{7}$$

$$\xrightarrow{N\to\infty} \quad \frac{(1-\gamma)^2}{4}\left[1-\frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] \tag{8}$$

(Proof in Appendix A.2.)

c) It holds that (see Hettmansperger (1984) pag. 89)

$$\frac{\sqrt{N}\left(\overline{CV}-\frac{(1-\gamma)}{2}\left[1-\frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]\right)}{\sqrt{\frac{(1-\gamma)^2}{4}\left[1-\frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]}} \tag{9}$$

converges to a standardized normal distribution (details in Appendix A.3.).

3. Acceptance breakdown of the proposed test.

Let V a test statistic which rejects the null hypothesis $H_0 : \theta = 0$ in favor of $H_1 : \theta > 0$ when $V \geq k$. The acceptance breakdown is defined as (See Hettmansperger & McKean (1998) pag 31):

$$\tau_N(\text{acceptance}) = \min\left\{\frac{m}{N} : \sup_{x}\inf_{x^{(m)}} V < k\right\} \tag{10}$$

where

$$x^{(m)} = (x_1^*, \ldots, x_m^*, x_{m+1}, \ldots, x_N)'$$

represents the corruption of m of the N observations. Denoting by a the minimum of the posible values of m wich satisfies (10), the acceptance breakdown is given by:

$$\tau_N(\text{acceptance}) = \frac{a}{N}.$$

That means V can tolerate at most a observations before it rejects H_0 and it can be controled with $a+1$ observations. The asymptotic acceptance breakdown ϵ can be approximated as solution of the equation (see Appendix A.4.):

$$\int_0^{1-\epsilon}\phi(u)du = \frac{1}{2}\int_0^1 \phi(u)du \tag{11}$$

The acceptance breakdown for the compared tests are the following (see proofs in Appendix A.4.1. and A.4.2.):

$$\text{S-Test:}\quad \epsilon_{\bar{S}} = 1/2, \quad \text{W-Test:}\quad \epsilon_{\bar{W}} = 1 - \sqrt{2}/2, \quad \text{B-Test:}\quad \epsilon_{B} = 1 - \left(\frac{1}{2}\right)^{\frac{1}{p+1}}$$

$$\text{WW-Test:}\quad \epsilon_{\bar{WW}} = \begin{cases} 1 - \sqrt{\frac{1-\gamma^2}{2}} & \text{if } \gamma < \frac{1}{3} \\ \frac{1+\gamma}{4} & \text{if } \gamma \geq \frac{1}{3} \end{cases} \tag{12}$$

$$\text{CV-Test:}\quad \epsilon_{\bar{CV}} = \begin{cases} 1 - \left[\frac{1-\gamma}{2}\left(1 + g(p) - g(p)(1-\gamma)^{\frac{1}{g(p)}}\right)\right]^{\frac{1}{g(p)+1}} & \text{if } \gamma \leq 1 - \left(\frac{g(p)+1}{g(p)+2}\right)^{g(p)} \\ \frac{1}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] & \text{if } \gamma > 1 - \left(\frac{g(p)+1}{g(p)+2}\right)^{g(p)}. \end{cases} \tag{13}$$

From equations (12) it is not difficult to see that:

1. $\epsilon_S > \epsilon_W$
2. $\epsilon_{WW} \to \epsilon_W$ if $\gamma \to 0$ and $\epsilon_{WW} \to \epsilon_S$ if $\gamma \to 1$
3. $\epsilon_B \to \epsilon_W$ when $p \to 1$ and $\epsilon_B \to \epsilon_S$ when $p \to 0$
4. $\epsilon_{\overline{CV}} \to \epsilon_{WW}$ if $g(p) \to 1$ and $\epsilon_{\overline{CV}} \to \epsilon_B$ when $\gamma \to 0$ for $g(p) = p$.
5. For $g(p) = p$ it holds $\epsilon_{\overline{CV}} \geq \epsilon_B$ for all $\gamma > 0$ y all p (see proof in appendix A.5.).

Let $\epsilon_{\overline{CV}}(g(p), \gamma)$, $\epsilon_{\overline{B}}(p)$ and $\epsilon_{\overline{WW}}(\gamma)$ be the acceptance breakdown of the correspondent tests as functions of the parameters p and γ. To ilustrate how $\epsilon_{\overline{CV}}$ for $g(p) = p$ overtakes ϵ_B, figure 1 shows the acceptance breakdown curves of the B and CV tests for $\gamma = 0, 0.25, 0.5, 0.75, 1$, between the acceptance breakdown curves of the W and the S tests as references. The isolines correspond to the WW test with the same values of γ.

It is convenient to clear that the acceptance breakdown of the W test and of the WW test are not comparable neither with the Baklizi nor with the proposed test because the first one are tests for samples coming from symmetrical distributions.

The acceptance breakdown of the CV-test tends to the acceptance breakdown of the WW-test when $p \to 1$. As expected, $\epsilon_{\overline{CV}}(p, 0)$ coincides with the $\epsilon_B(p)$. Moreover, for each p, $\epsilon_{\overline{CV}}(p, \gamma)$ increases with γ up to $1/2$ (the acceptance breakdown of the S-test) for $\gamma = 1$. That means that for each p, to get greater acceptance breakdown for the CV-test, more observations must be trimmed. However, the closer to one is p the larger is the number of trimmed observations required to get larger acceptance breakdown.

It is posible to increase or decrease the acceptance breakdown of the CV-test with other selections of $g(p)$. For example with $g(p) = p^{1/k}$, $k = 1, 2, 3, 5, 10$,

$\epsilon_{\overline{CV}}(p^{1/k}, \gamma)$ decreases when k increases for γ close to zero, and for all values of p; it tends also to the acceptance breakdown of the W-test (see figure 2 for $\gamma = 0$).

To increase the acceptance breakdown of the CV-test so that the CV-test is more tolerant than the WW-test, $g(p)$ can be selected for example as $g(p) = 1/2$. Then the acceptance breakdown of the CV-test is much more greater than the acceptance breakdown of the WW-test for $p = 1$ and all γ (see figure 3). On the other hand, to decrease the acceptance breakdown of the CV-test so that the CV-test is less tolerant than the WW-test, $g(p)$ can be selected for example as $g(p) = \sqrt{2}$. (see figure 4)

To increase the acceptance breakdown of the CV-test so that the CV-test is more tolerant than the B-test, $g(p)$ can be selected for example as $g(p) = ln(p+1)$. Then the acceptance breakdown of the CV-test is much more greater than the acceptance breakdown of the B-test for all p and all γ (see figure 5). Moreover the acceptance breakdown of the CV-test overtakes the acceptance breakdown of the WW-test when $p = 1$ and for all γ.

4. Conclusions and discussion

The selection of $\phi(u) = \min\{u^{\frac{1}{2}}, 1 - \gamma\}$ as a score function instead of $\phi(u) = \min\{u, 1-\gamma\}$ produces a more tolerant test to bad observations than the WW-test for all values of γ. In other words, with the same proportion of reject observations and such an selection of φ, the proposed test has bigger acceptance breakdown that the WW-test.

The acceptance breakdown to the null hypothesis of the proposed test is bigger than the acceptance breakdown of the Baklizi test for all values of p with $g(p) = p$. The proposed test needs no reject observations to have bigger acceptance breakdown that the Baklizi test.

For $g(p) = p$ and $g(p) = \ln(p + 1)$ it holds following: the bigger is the asymmetry of the sampled distribution (smaller p-value), the bigger is the acceptance breakdown of the proposed test. The bigger is the proportion of rejected observations, bigger is the acceptance breakdown of the proposed test.

The proposed test as generalization of both the WW-and the B-test permits to chose $g(p)$ functions, such that it can be more, or less tolerante to acceptance than its competitors.

Discussion

There are still two questions about de use of the proposed test: the first one about the power of the test, the second one about the use of the test statistic to estimate the location parameter of the sampled distribution. To the first one it must be estimated the power of the CV-test for some selections of the $g(p)$ function and of the proportion of winsorized observations γ, such that it can be produce powerful tests. To the second one question it will be interesting to explore some other properties of the CV statistic as the influence function.

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Appendix A. Proofs of the main results.

Appendix A.1. Validity conditions for the score function.

The function $\phi(u) = \min\{u^{g(p)}, 1-\gamma\}$ is a well defined score function.

Proof. ϕ is a nonnegative function since $0 < u < 1$, $0 < g(p) \leq 1$ and $0 \leq 1-\gamma \leq 1$. Now ϕ is nondecreasing because:

$$\phi'(u) = \begin{cases} g(p)u^{(g(p)-1)} & \text{si } u \leq (1-\gamma)^{\frac{1}{g(p)}} \\ 0 & \text{elsewhere} \end{cases}$$

is nonnegative. On the other hand

$$\begin{aligned}
\int_0^1 \phi(u)\,du &= \int_0^1 \min\{u^{g(p)}, 1-\gamma\}\,du \\
&= \int_0^{(1-\gamma)^{\frac{1}{g(p)}}} u^{g(p)}\,du + \int_{(1-\gamma)^{\frac{1}{g(p)}}}^1 1-\gamma\,du \\
&= \frac{(1-\gamma)^{1+\frac{1}{g(p)}}}{g(p)+1} + (1-\gamma)\left(1-(1-\gamma)^{\frac{1}{g(p)}}\right) \\
&= (1-\gamma)\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right].
\end{aligned}$$

Since $0 < g(p) < 1$ implies $0 < \frac{g(p)}{g(p)+1} < 1$, and $0 < 1-\gamma < 1$ implies $0 < (1-\gamma)^{\frac{1}{g(p)}} < 1$.

Then $0 < 1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}} < 1$. Hence, $\int_0^1 \min\{u^{g(p)}, 1-\gamma\}\, du < 1-\gamma$.

The last condition for ϕ follows because of

$$\begin{aligned}
\int_0^1 \phi^2(u)\, du &= \int_0^1 \min{}^2\{u^{g(p)}, 1-\gamma\}\, du \\
&= \int_0^{(1-\gamma)^{\frac{1}{g(p)}}} u^{2\,g(p)}\, du + \int_{(1-\gamma)^{\frac{1}{g(p)}}}^1 (1-\gamma)^2\, du \\
&= \frac{(1-\gamma)^{2+\frac{1}{g(p)}}}{2\,g(p)+1} + (1-\gamma)^2 \left(1 - (1-\gamma)^{\frac{1}{g(p)}}\right) \\
&= (1-\gamma)^2 \left[1 - \frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]
\end{aligned}$$

Finally

$$0 < \frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}} < \frac{2}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}$$

$$0 < 1 - \frac{2}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}} < 1 - \frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}} < 1$$

implies

$$\left(1 - \frac{2}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right)(1-\gamma)^2 < \int_0^1 \min{}^2\{u^{g(p)}, 1-\gamma\}\, du < (1-\gamma)^2$$

□

Appendix A.2. Conditional mean and variance of the proposed test statistic under H_0.

Proof.

$$\begin{aligned}
E\left[\,\overline{CV} \mid p\,\right] &= E\left[\frac{1}{N}\sum_{i=1}^{N} \min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} s(X_i) \;\middle|\; p\right] \\
&= \frac{1}{N}\sum_{i=1}^{N} \min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} E\left[\, s(X_i) \mid p\,\right] \\
&= \frac{1}{2}\sum_{i=1}^{N} \min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)}, 1-\gamma\right\} \frac{1}{N} \\
&\xrightarrow{N\to\infty} \frac{1}{2}\int_0^1 \min\{u^{g(p)}, 1-\gamma\}\, du \\
&= \frac{(1-\gamma)}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]
\end{aligned}$$

$$
\begin{aligned}
N\,Var\,[\,\overline{CV}\mid p\,] &= N\,Var\left[\frac{1}{N}\sum_{i=1}^{N}\min\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)},1-\gamma\right\}s(X_i)\;\middle|\;p\,\right]\\
&= \frac{N}{N^2}\sum_{i=1}^{N}\min^2\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)},1-\gamma\right\}Var\,[\,s(X_i)\,|\,p\,]\\
&= \frac{1}{4}\sum_{i=1}^{N}\min^2\left\{\left(\tfrac{R_i}{N+1}\right)^{g(p)},1-\gamma\right\}\frac{1}{N}\\
&\xrightarrow{N\to\infty}\quad \frac{1}{4}\int_0^1 \min^2\{u^{g(p)},1-\gamma\}\,du\\
&= \frac{(1-\gamma)^2}{4}\left[1-\frac{2g(p)}{2g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]
\end{aligned}
$$

□

Appendix A.3. Noether condition for the score function.

$$
\frac{\max\limits_{1\le i\le N}\phi\left(\frac{i}{N+1}\right)}{\sqrt{\sum_{i=1}^{N}\phi^2\left(\frac{i}{N+1}\right)}}\xrightarrow{N\to\infty}0 \tag{14}
$$

***Proof*.** As ϕ it is nonnegative and nondecreasing, ϕ^2 also has these properties. Then from (14)

$$
\frac{\phi^2\left(\frac{N}{N+1}\right)}{\sum_{i=1}^{N}\phi^2\left(\frac{i}{N+1}\right)}=\frac{\frac{1}{N+1}\min^2\left\{\left(\frac{N}{N+1}\right)^{g(p)},1-\gamma\right\}}{\frac{1}{N+1}\sum_{i=1}^{N}\min^2\left\{\left(\frac{i}{N+1}\right)^{g(p)},1-\gamma\right\}}
$$

Note that

$$
\lim_{N\to\infty}\frac{1}{N+1}\sum_{i=1}^{N}\min^2\left\{\left(\frac{i}{N+1}\right)^{g(p)},1-\gamma\right\}\longrightarrow\int_0^1\min^2\{u^{g(p)},1-\gamma\}\,du<(1-\gamma)^2 \tag{15}
$$

Then is enough to proof that the numerator converges to 0. Given that $\phi(u)$ is nondecreasing follows

$$
0\le\tfrac{1}{N+1}\min^2\left\{\left(\tfrac{N}{N+1}\right)^{g(p)},1-\gamma\right\}\le\int_{\frac{N}{N+1}}^{1}\min^2\{u^{g(p)},1-\gamma\}\,du
$$

If $\frac{N}{N+1}>(1-\gamma)^{\frac{1}{g(p)}}$, then

$$
\int_{\frac{N}{N+1}}^{1}\min^2\{u^{g(p)},1-\gamma\}\,du=\int_{\frac{N}{N+1}}^{1}(1-\gamma)^2\,du=(1-\gamma)^2\left(1-\tfrac{N}{N+1}\right)\xrightarrow{N\to\infty}\;0.
$$

If $\frac{N}{N+1} \leq (1-\gamma)^{\frac{1}{g(p)}}$, then

$$\int_{\frac{N}{N+1}}^{1} \min^2\{u^{g(p)}, 1-\gamma\}\, du = \int_{\frac{N}{N+1}}^{(1-\gamma)^{\frac{1}{g(p)}}} u^{2g(p)}\, du + \int_{(1-\gamma)^{\frac{1}{g(p)}}}^{1} (1-\gamma)^2\, du$$

$$= (1-\gamma)^2 \left[1 - \frac{2g(p)}{2g(p)+1}\,(1-\gamma)^{\frac{1}{g(p)}}\right] - \frac{\left(\frac{N}{N+1}\right)^{2g(p)+1}}{2g(p)+1}$$

by hypothesis $\frac{N}{N+1} \leq (1-\gamma)^{\frac{1}{g(p)}}$ y $0 < (1-\gamma)^2 < 1$, implies

$$\int_{\frac{N}{N+1}}^{1} \min^2\{u^{g(p)}, 1-\gamma\}\, du \leq (1-\gamma)^2 \left[1 - \frac{2g(p)}{2g(p)+1}\left(\frac{N}{N+1}\right)\right] - \frac{\left(\frac{N}{N+1}\right)^{2g(p)+1}}{2g(p)+1}$$

$$\leq 1 - \frac{2g(p)}{2g(p)+1}\left(\frac{N}{N+1}\right) - \frac{\left(\frac{N}{N+1}\right)^{2g(p)+1}}{2g(p)+1}$$

$$\xrightarrow{N\to\infty} \quad 1 - \frac{2g(p)}{2g(p)+1} - \frac{1}{2g(p)+1} = 0.$$

□

Appendix A.4. Acceptance breakdown of the test.

To evaluate the acceptance breakdown of the componed tests use the following approximations given in (Hettmansperger 1984, p. 90-91):

$$\int_0^{1-\epsilon} \phi(u)du = \frac{1}{2}\int_0^1 \phi(u)du \tag{16}$$

Acceptance breakdown of the S, W and WW tests are in Hettmansperger & McKean (1998) calculated.

Appendix A.4.1. Acceptance breakdown of the Baklizi test.

***Proof*.** The asymptotic acceptance breakdown to acceptance of the Baklizi test is defined by (16). Then

$$\int_0^{1-\epsilon} u^p\, du = \frac{1}{2}\int_0^1 \phi(u)du = \frac{1}{2p+2}$$

Then $(1-\epsilon)^{p+1} = \frac{1}{2}$ and

$$\epsilon = 1 - \left(\frac{1}{2}\right)^{\frac{1}{p+1}}.$$

□

Appendix A.4.2. Acceptance breakdown of the proposed test.

Proof. The asymptotic acceptance breakdown of the proposed test is ϵ, defined by (16)

$$\int_0^{1-\epsilon} \min\{u^{g(p)}, 1-\gamma\}\, du = \frac{1-\gamma}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]$$

If $1-\epsilon \leq (1-\gamma)^{\frac{1}{g(p)}}$, then

$$\begin{aligned}
\int_0^{1-\epsilon} u^{g(p)}\, du &= \frac{1-\gamma}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] \\
(1-\epsilon)^{g(p)+1} &= \frac{1-\gamma}{2}\left[1 + g(p) - g(p)(1-\gamma)^{\frac{1}{g(p)}}\right] \\
\epsilon &= 1 - \left[\frac{1-\gamma}{2}\left(1 + g(p) - g(p)(1-\gamma)^{\frac{1}{g(p)}}\right)\right]^{\frac{1}{g(p)+1}}
\end{aligned}$$

by hypothesis

$$\begin{aligned}
1-\epsilon &\leq (1-\gamma)^{\frac{1}{g(p)}} \\
(1-\epsilon)^{g(p)+1} &\leq (1-\gamma)^{1+\frac{1}{g(p)}} \\
\frac{1-\gamma}{2}\left[1 + g(p) - g(p)(1-\gamma)^{\frac{1}{g(p)}}\right] &\leq (1-\gamma)^{1+\frac{1}{g(p)}} \\
\frac{g(p)+1}{g(p)+2} &\leq (1-\gamma)^{\frac{1}{g(p)}} \\
\gamma &\leq 1 - \left(\frac{g(p)+1}{g(p)+2}\right)^{g(p)}
\end{aligned}$$

If $1-\epsilon > (1-\gamma)^{\frac{1}{g(p)}}$, then

$$\begin{aligned}
\int_0^{(1-\gamma)^{\frac{1}{g(p)}}} u^{g(p)}\, du + \int_{(1-\gamma)^{\frac{1}{g(p)}}}^{1-\epsilon} 1-\gamma\, du &= \frac{1-\gamma}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] \\
(1-\gamma)\left[\frac{(1-\gamma)^{\frac{1}{g(p)}}}{g(p)+1} + (1-\epsilon) - (1-\gamma)^{\frac{1}{g(p)}}\right] &= \frac{1-\gamma}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] \\
1-\epsilon &= \frac{1}{2}\left[1 + \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] \\
\epsilon &= \frac{1}{2}\left[1 - \frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right]
\end{aligned}$$

by hypothesis

$$
\begin{aligned}
(1-\gamma)^{\frac{1}{g(p)}} &< 1-\epsilon \\
2\,(1-\gamma)^{\frac{1}{g(p)}} &< 1+\frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}} \\
(1-\gamma)^{\frac{1}{g(p)}} &< \frac{g(p)+1}{g(p)+2} \\
\gamma &> 1-\left(\frac{g(p)+1}{g(p)+2}\right)^{g(p)}
\end{aligned}
$$

Hence the asymptotic acceptance breakdown of the proposed test is

$$
\epsilon=\begin{cases} 1-\left[\frac{1-\gamma}{2}\left(1+g(p)-g(p)(1-\gamma)^{\frac{1}{g(p)}}\right)\right]^{\frac{1}{g(p)+1}} & \text{si} \quad \gamma\leq 1-\left(\frac{g(p)+1}{g(p)+2}\right)^{g(p)} \\ \frac{1}{2}\left[1-\frac{g(p)}{g(p)+1}(1-\gamma)^{\frac{1}{g(p)}}\right] & \text{si} \quad \gamma> 1-\left(\frac{g(p)+1}{g(p)+2}\right)^{g(p)}. \end{cases}
$$

□

Appendix A.5. Comparing the asymptotic acceptance breakdown of $\bar{CV}$-test and $\bar{B}$-test.

Proof. Taking $g(p)=p$ in (13) implies $0<\gamma\leq 1/3$ when $p\to 1$ for the first case and $1/3<\gamma<1$ in the second case. Then for $0<p<1$ it holds:

$$
\epsilon_{\overline{CV}}=\begin{cases} 1-\left[\frac{1-\gamma}{2}\left(1+p-p(1-\gamma)^{\frac{1}{p}}\right)\right]^{\frac{1}{p+1}} & \text{for} \quad 0<\gamma\leq 1/3 \\ \frac{1}{2}\left[1-\frac{p}{p+1}(1-\gamma)^{\frac{1}{p}}\right] & \text{for} \quad 1/3<\gamma<1. \end{cases} \tag{17}
$$

For the first case when $0<\gamma\leq 1/3$:
if $p\to 0$ $\epsilon_{\overline{CV}}\to\frac{1+\gamma}{2}\geq 1/2=\epsilon_B$. When $p\to 1$, $\epsilon_{\overline{CV}}\to 1-\left(\frac{1-\gamma^2}{2}\right)^{(1/2)}$, which goes to $1-\frac{1}{\sqrt{2}}=\epsilon_B$ when $\gamma\to 0$, and for $\gamma\to 1$, $\epsilon_{\overline{CV}}\to 1>\epsilon_B$.
In the second case, $1/3<\gamma\leq 1$:
when $p\to 0$, $\epsilon_{\overline{CV}}\to 1/2=\epsilon_B$, and when $p\to 1$, $\epsilon_{\overline{CV}}\to\frac{1}{4}(1+\gamma)$, which goes to $1/3>\epsilon_B$ when $\gamma\to 1/3$ and goes to $1/2>\epsilon_B$ for $\gamma\to 1$. □

FIGURE 1: Acceptance breakdown of the $\bar{B}$-test and $\bar{CV}$-test for $g(p) = p$.

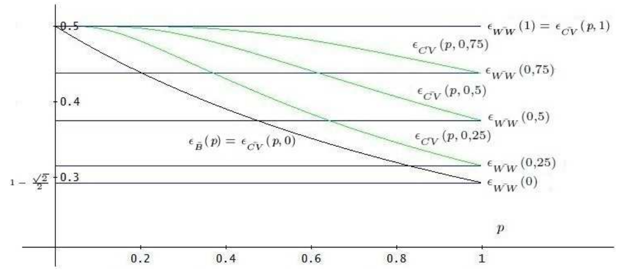

FIGURE 2: Acceptance breakdown of the $\bar{B}$-test and $\bar{CV}$-test for $g(p) = p^{1/k}$.

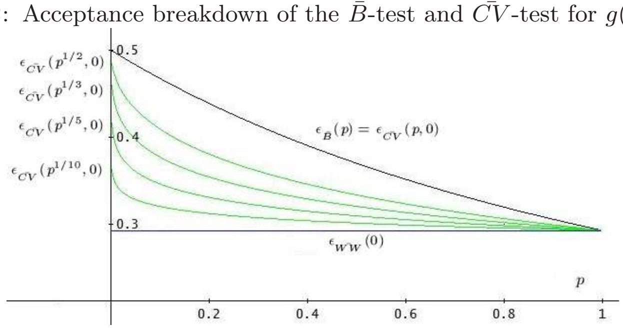

FIGURE 3: Acceptance breakdown of the CV-test and WW-test for $g(p) = 1/2$.

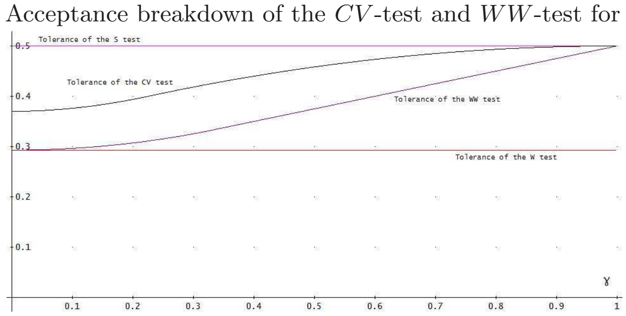

FIGURE 4: Acceptance breakdown of the CV-test and WW-test for $g(p) = \sqrt{2}$.

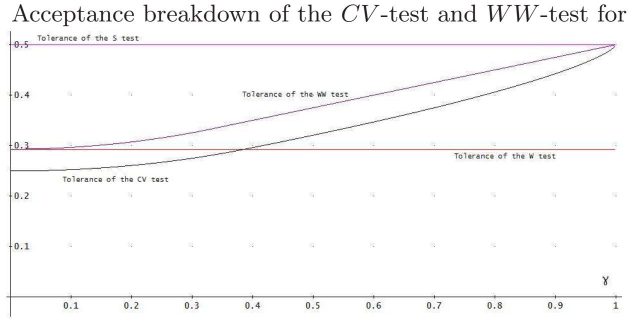

FIGURE 5: Acceptance breakdown of the $\bar{B}$ and $\bar{CV}$-tests for $g(p) = ln(p+1)$.