The Beta Moyal: An Useful Skew Distribution

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Abstract

For the first time, we propose a so-called beta Moyal distribution, which generalizes the Moyal distribution, and study its properties. We derive expansions for the cumulative distribution function as a weighted power series of the Moyal cumulative distribution. We provide a comprehensive mathematical treatment of the new model and derive expansions for its moments, moment generating function, mean deviations, density function of the order statistics and their moments. We discuss maximum likelihood estimation of the model parameters. We illustrate the superiority of the new distribution as compared to the beta normal, skew-normal and Moyal distributions by means of two real data sets.

Keywords: Expected information; Hazard rate function; Maximum likelihood estimation; Moyal distribution; Survival function.

1 Introduction

One major benefit of the class of beta generalized distributions proposed by Eugene *et al.* (2002) is its ability of fitting skewed data that can not be properly fitted by existing distributions. Starting from a parent cumulative distribution function (cdf) $G(x)$, this class is defined by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1} d\omega, \tag{1}$$

where a and b are additional positive parameters, $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(a)$ is the gamma function. This class of generalized distributions has been receiving considerable attention over the last years in particular after the work of Jones (2004).

Eugene *et al.* (2002), Nadarajah and Kotz (2004), Nadarajah and Gupta (2004), Nadarajah and Kotz (2005), Lee *et al.* (2007) and Akinsete *et al.* (2008) defined the beta normal, beta Gumbel, beta Fréchet, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cdf of the normal, Gumbel, Fréchet, exponential, Weibull and Pareto distributions, respectively. More recently, Barreto-Souza *et al.* (2009) and Pescim *et al.* (2010) introduced the beta generalized exponential and the beta generalized half-normal distributions, respectively.

The probability density function (pdf) corresponding to (1) is

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\{1-G(x)\}^{b-1}, \tag{2}$$

where $g(x) = dG(x)/dx$ is the parent density function. The density $f(x)$ will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions. Except for some special choices of these functions, $f(x)$ will be difficult to deal with some generality.

In this article, we introduce a four parameter distribution, so-called the beta Moyal (BMo) distribution, which contains as a sub-model the Moyal distribution. The BMo distribution is convenient for modeling comfortable upside-down bathtub-shaped failure rates and as a competitive model to the Moyal, half normal, beta normal, skew normal and Gumbel distributions.

The article is organized as follows. In Section 2, we define the BMo distribution, present some special sub-models and provide expansions for its distribution and density functions. Section 3 gives general expansions for the moments, moment generating function (mgf), mean deviations and Rényi entropy. In Section 4, we derive expansions for the moments of order statistics. Maximum likelihood estimation and inference issues are addressed in Section 5. Section 6 illustrates the importance of the BMo distribution through two real data sets. Finally, concluding remarks are given in Section 7.

2 Beta Moyal Distribution

The Moyal distribution was proposed for J. E. Moyal (1955) as an approximation for the Landau distribution. It was also shown that it remains valid taking into account quantum resonance effects and details of atomic structure of the absorber. Let X be a random variable following the Moyal standard density function given by

$$g_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-[x+\exp(-x)]/2\}, \quad -\infty < x < \infty. \tag{3}$$

A location parameter μ and a scale factor σ can be introduced to define the random variable $Z = \sigma X + \mu$ having a Moyal distribution, say $\mathrm{Mo}(\mu,\sigma)$, given by

$$g_Z(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma}\right) + \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}\right]\right\}, \tag{4}$$

where $-\infty < x, \mu < \infty$ and $\sigma > 0$. The cumulative distribution function (cdf) corresponding to (4) is

$$G_Z(x) = 1 - \frac{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma(\frac{1}{2})}, \tag{5}$$

which depends on the incomplete gamma function $\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}dt$.

The Moyal distribution is a universal form for the energy loss by ionization for a fast charged particle and the number of ion pairs produced in this process. The cumulants of the standard Moyal distribution (3) are

$$\kappa_1 = -\log(2) - \psi\Big(\frac{1}{2}\Big) = \log(2) + \gamma \text{ and } \kappa_n = (-1)^n \psi^{n-1}\Big(\frac{1}{2}\Big) = (n-1)!(2^n-1)\zeta(n),\ n \geq 2,$$

where $\gamma \approx 0.57721$ is Euler's constant, $\psi^{(n)}$ denotes polygamma functions and $\zeta(.)$ is the Riemann's zeta function defined by

$$\zeta(u) = \sum_{k=1}^{\infty} \frac{1}{k^u} = \frac{1}{\Gamma(u)} \int_0^{\infty} \frac{x^{u-1}}{e^x - 1} dx \quad \text{for} \quad u > 1.$$

The moments can be easily obtained from these cumulants. Those of lower order are $\mu_1' = E(X) = \log(2) + \gamma \approx 1.27036$, $\mu_2' = Var(X) = \frac{\pi^2}{2} \approx 4.9348$, $\mu_3' = 14\zeta_3$ and $\mu_4' = \frac{7\pi^4}{4}$. For the distribution (4), $E(Z) = \sigma E(X) + \mu$, $Var(Z) = \sigma^2 Var(X)$ and, more generally, the central moments of Z ($\mu_{n,Z}$) are easily obtained from the central moments of X ($\mu_{n,X}$) by $\mu_{n,Z} = \sigma^n \mu_{n,X}$ for $n \geq 2$.

The characteristic function of the Moyal distribution (3) is

$$\phi(t) = E(e^{itz}) = \frac{2^{-it}}{\sqrt{\pi}} \Gamma\Big(\frac{1}{2} - \text{it}\Big),$$

where the gamma function $\Gamma(.)$ with complex argument is valid when the real part of the argument is positive which is indeed true in the case at hand.

The Moyal distribution can be defined in a finite interval. For this purpose, the transformation $X = \tan(Y)$ yields the density function of Y as

$$\pi(y) = \frac{1}{\sqrt{2\pi}\cos^2(y)} \exp\{-\frac{1}{2}\big[\tan(y) + \exp\{-\tan(y)\}\big]\}, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

This density function has a maximum of about 0.91 and it is widely used to generate Moyal variates.

We now introduce the four parameter BMo distribution by taking $G(x)$ in (1) to be the cdf (5) of the Moyal distribution. The BMo cumulative function is given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{1 - \frac{\gamma\{\frac{1}{2}, \frac{1}{2}\exp[-(\frac{x-\mu}{\sigma})]\}}{\Gamma(\frac{1}{2})}} \omega^{a-1}(1-\omega)^{b-1} d\omega. \quad (6)$$

Inserting (4) and (5) into (2) yields the BMo density function

$$f(x) = \frac{\exp\{-\frac{1}{2}[(\frac{x-\mu}{\sigma}) + \exp\{-(\frac{x-\mu}{\sigma})\}]\}}{\sqrt{2\pi}\sigma B(a,b)\Gamma(\frac{1}{2})^{b-1}} \left\{1 - \frac{\gamma\{\frac{1}{2}, \frac{1}{2}\exp[-(\frac{x-\mu}{\sigma})]\}}{\Gamma(\frac{1}{2})}\right\}^{a-1} \times$$
$$\left\{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\right\}^{b-1}, \quad -\infty < x < \infty, \quad (7)$$

where $-\infty < \mu < \infty$ is the location parameter, $\sigma > 0$ is the scale parameter and $a > 0$ and $b > 0$ are shape parameters. For $a = b = 1$, it reduces to the Moyal distribution. For $\mu = 0$ and $\sigma = 1$, we obtain the standard BMo density function given by

$$f(x) = \frac{\exp\{-\frac{1}{2}[x + \exp(-x)]\}}{\sqrt{2\pi} B(a,b)\Gamma(\frac{1}{2})^{b-1}} \left\{1 - \frac{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]}{\Gamma(\frac{1}{2})}\right\}^{a-1} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^{b-1}. \quad (8)$$

Plots of the density function (8) are illustrated in Figure 1 for selected parameter values, including the special case of the standard Moyal distribution.

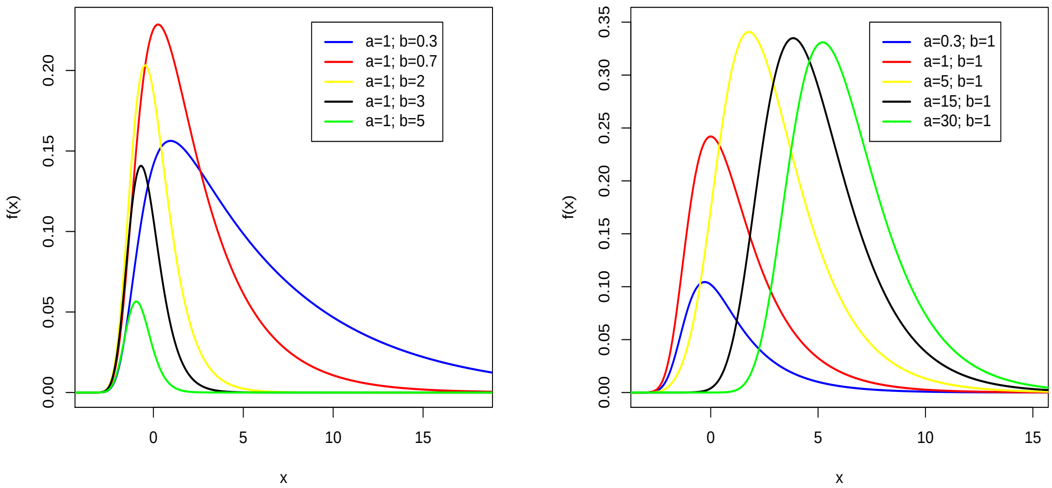

Figure 1: Plots of the density function (8) for some parameter values.

The hazard rate function corresponding to (7) becomes

$$h(x) = \frac{\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma}\right)+\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}\right]\right\}\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma\left(\frac{1}{2}\right)}\right\}^{a-1}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{b-1}B(a,b)\left\{1-I_{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma\left(\frac{1}{2}\right)}}(a,b)\right\}\left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\right\}^{-(b-1)}}. \tag{9}$$

Plots of some standard BMo hazard rate functions for selected parameter values are given in Figure 2.

If X is a random variable with density function (7), we write $X \sim$BMo(a,b,μ,σ). The BMo distribution is easily simulated from $F(x)$ in (6) as follows: if V has a beta distribution with parameters a and b, then the solution of the nonlinear equation

$$\frac{X-\mu}{\sigma} = -\log\left\{2\left[\text{erf}^{-1}(1-V)\right]^2\right\}$$

yields the BMo(a,b,μ,σ) distribution, where $\gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\,\text{erf}(\sqrt{x})$ and the error function erf(x) is defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. To simulate data from this nonlinear equation, we can use the matrix programming language Ox through the *SolveNLE* subroutine (see Doornik, 2007).

We provide two simple formulae for the cdf of the BMo distribution depending if the parameter $b>0$ is real non-integer or integer. First, if $|z|<1$ and $b>0$ is real non-integer, we have the series representation

$$(1-z)^{b-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}z^j. \tag{10}$$

For $b>0$ real non-integer, using the representation (10), the standard cumulative function (6) (for $\mu=0$ and $\sigma=1$) can be expanded as

$$F(x) = \frac{\Gamma(b)}{B(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\left\{1-\frac{\gamma\left(\frac{1}{2},\frac{\exp(-x)}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right\}^{a+j}}{\Gamma(b-j)j!(a+j)}. \tag{11}$$

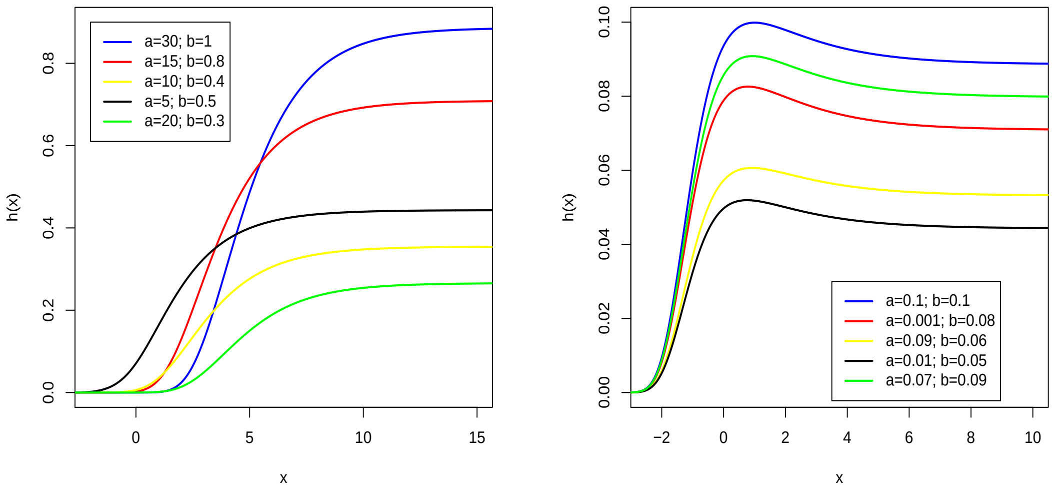

Figure 2: Plots of the hazard rate function (9) for some parameter values and $\mu = 0$ and $\sigma = 1$. (a) Increasing (b) Upside-down bathtub.

If $a > 0$ is an integer, (11) gives the cdf of the BMo distribution in terms of an infinite weighted power series of the Moyal cumulative function. Otherwise, if $a > 0$ is real non-integer, the expansion (10) in equation (11) yields

$$F(x) = \frac{\Gamma(b)}{B(a,b)} \sum_{j,r=0}^{\infty} \frac{(-1)^{j+r} \gamma\left(\frac{1}{2}, \frac{\exp(-x)}{2}\right)^r}{\Gamma(b-j)\Gamma(a+j+1-r)\Gamma(\frac{1}{2})^r j! r!}. \quad (12)$$

Equation (12) shows that for both b and a real non-integers, the cdf of the BMo distribution can be expressed as an infinite weighted sum of powers of the incomplete gamma function.

By application of the binomial expansion in equation (6), when $b > 0$ is an integer, we obtain

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{a+j} \left\{ 1 - \frac{\gamma\left(\frac{1}{2}, \frac{\exp(-x)}{2}\right)}{\Gamma(\frac{1}{2})} \right\}^{a+j}. \quad (13)$$

For $a > 0$ integer, using the binomial expansion in (13), yields

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \sum_{r=0}^{a+j} \frac{(-1)^{j+r}}{(a+j)\Gamma(\frac{1}{2})^r} \binom{b-1}{j} \binom{a+j}{r} \gamma\left(\frac{1}{2}, \frac{\exp(-x)}{2}\right)^r. \quad (14)$$

For $a > 0$ real non-integer, expanding (13) as in (10), we have

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \sum_{r=0}^{\infty} \frac{(-1)^{j+r} 2^r \Gamma(a+j+1)}{(a+j)\Gamma(a+j+1-r)\Gamma(\frac{1}{2})^r r!} \binom{b-1}{j} \binom{r}{s} \gamma\left(\frac{1}{2}, \frac{\exp(-x)}{2}\right)^r. \quad (15)$$

The standard Moyal cdf can be obtained from equation (13) when $a = b = 1$. Equations (11)-(15) are the main expansions for the cdf of the BMo distribution. They (and other expansions in the paper) can be evaluated in symbolic computation software such as Mathematica and Maple. These symbolic software have currently the ability to deal with analytic expressions of formidable size and complexity.

Alternatively to (8), an expansion for the standard BMo density function for b real non-integer follows by differentiating (11) and using the series representation (10)

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}} \exp\{-[x+\exp(-x)]/2\} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) \left\{\gamma\left(\frac{1}{2}, \frac{\exp(-x)}{2}\right)\right\}^{k}, \tag{16}$$

where the coefficients $w_{j,k}(a,b)$ are given by

$$w_{j,k}(a,b) = \frac{(-1)^{j+k}\Gamma(b)\Gamma(a+j)}{\Gamma(b-j)\Gamma(a+j-k)\Gamma(\frac{1}{2})^{k} k! j! B(a,b)}.$$

Equation (16) is the basic expansion for the standard BMo density function.

3 Properties of the beta Moyal distribution

We hardly need to emphasize the necessity and importance of moments and mgf in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

3.1 Moments

Theorem 1: If $X \sim \mathrm{BMo}(a,b,0,1)$, the sth moment of X is given by

$$\mu_s' \quad = \quad \frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} \sum_{r=0}^{s} \sum_{m=0}^{\infty} v_{j,k,r,s,m}(a,b)\Gamma_r\left(m+\frac{k+1}{2}\right), \tag{17}$$

where all quantities are defined in the following proof.

Proof:

The sth moment of the BMo distribution is $\mu_s' = \int_{-\infty}^{\infty} x^s f(x)dx$. Hence, if $b>0$ is real non-integer, we obtain from (16)

$$\mu_s' \quad = \quad \frac{1}{\sqrt{2\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) \int_{-\infty}^{\infty} x^s \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}\left[x+\exp(-x)\right]\} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^{k} dx.$$

Setting $u = \exp(-x)/2$, μ_s' reduces to

$$\mu_s' = \frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b)(-1)^s \int_0^{\infty} \log^s(2u)\, u^{-\frac{1}{2}}\, e^{-u} \left[\gamma\left(\frac{1}{2}, u\right)\right]^{k} du.$$

Using the binomial expansion in the last equation, we can obtain

$$\mu_s' = \frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} \sum_{r=0}^{s} w_{j,k}(a,b)(-1)^s \binom{s}{r} \log^{(s-r)}(2) \int_0^{\infty} \log^r(u)\, u^{-\frac{1}{2}}\, e^{-u} \left[\gamma\left(\frac{1}{2}, u\right)\right]^{k} du. \tag{18}$$

Using the series expansion $\gamma(\alpha,x) = x^{\alpha} \sum_{m=0}^{\infty} \frac{(-x)^m}{(\alpha+m)m!}$, we can rewrite the integral in (18), say $I(r,k)$, using the identity of a power series raised to an integer, namely $(\sum_{k=0}^{\infty} a_k x^k)^n = \sum_{k=0}^{\infty} c_{k,n} x^k$ (see Gradshteyn and Ryzhik, 2000), where $c_{0,n} = a_0^n$ and $c_{k,n} = (k a_0)^{-1} \sum_{l=1}^{k} (nl-k+l) a_l c_{k-l,n}$. Hence,

$$I(r,k) = \int_0^{\infty} \log^r(u)\, u^{-\frac{1}{2}}\, e^{-u} \left[u^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-u)^m}{(\frac{1}{2}+m)m!}\right]^{k} du = \int_0^{\infty} \log^r(u)\, u^{\frac{k-1}{2}}\, e^{-u} \sum_{m=0}^{\infty} c_{m,k} u^m du,$$

where $c_{m,k} = \frac{1}{m}\sum_{l=1}^{m}\frac{(-1)^l(kl-m+l)}{(2l+1)l!}c_{m-l,k}$ for $m = 1, 2, \ldots$ and $c_{0,k} = 2^k$, $k = 1, 2, \ldots$ Inserting the last equation in (18) yields

$$\mu_s' = \frac{1}{\sqrt{\pi}}\sum_{j,k=0}^{\infty}\sum_{r=0}^{s}\sum_{m=0}^{\infty} w_{j,k}(a,b)(-1)^s\binom{s}{r}\log^{(s-r)}(2)c_{m,k}\int_0^\infty \log^r(u)\; u^{m+\frac{k+1}{2}-1}\; e^{-u}du. \tag{19}$$

The integral $J(r)$ in (19) can be easily calculated from the result given by Prudnikov *et al.* (1986, Vol.1, Section 2.6.21, integral 1). From the definition of $\Gamma_r(p) = \frac{\partial^r\Gamma(p)}{\partial p^r}$, we have

$$J(r) = \int_0^\infty \log^r(u)\; u^{m+\frac{k+1}{2}-1}\; e^{-u}du = \Gamma_r\left(m + \frac{k+1}{2}\right).$$

Hence, the sth moment of the standard BMo distribution can be expressed as

$$\mu_s' \;=\; \frac{1}{\sqrt{\pi}}\sum_{j,k=0}^{\infty}\sum_{r=0}^{s}\sum_{m=0}^{\infty} v_{j,k,r,s,m}(a,b)\Gamma_r\left(m+\frac{k+1}{2}\right),$$

where

$$v_{j,k,r,s,m}(a,b) = w_{j,k}(a,b)(-1)^s\binom{s}{r}\log^{(s-r)}(2)c_{m,k}. \quad \blacksquare$$

The skewness and kurtosis measures can now be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis for some choices of the parameter b as function of a, and for some choices of the parameter a as function of b, for $\mu = 0$ and $\sigma = 1$, are shown in Figures 3 and 4, respectively.

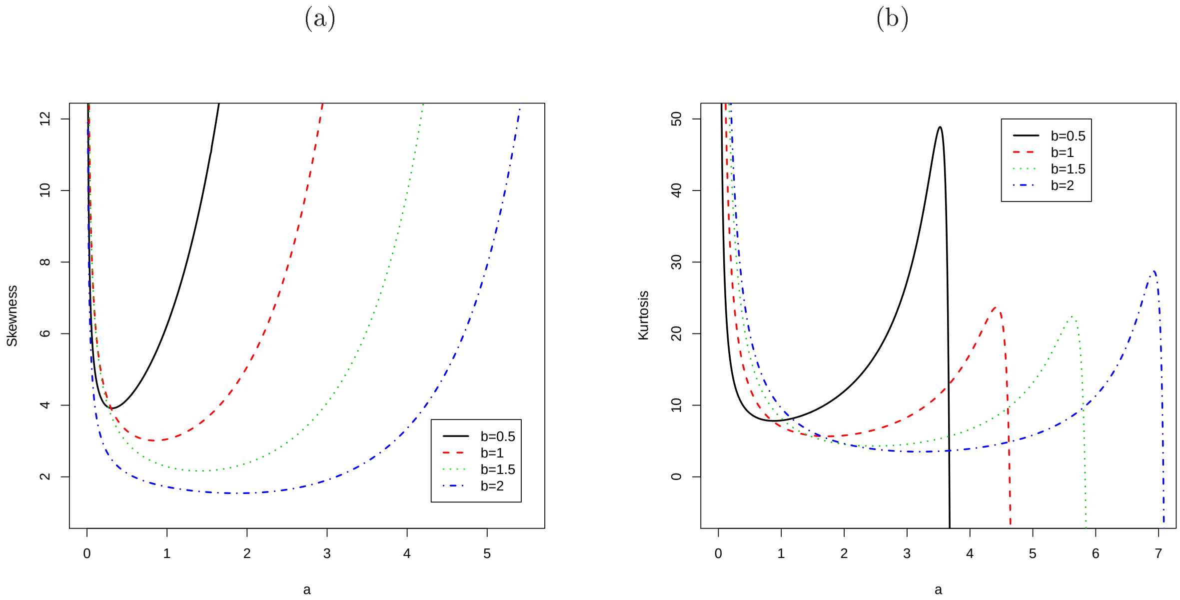

Figure 3: Skewness and kurtosis of the standard BMo distribution as a function of a for selected values of b.

3.2 Moment generating function

Theorem 2: If $X \sim \mathrm{BMo}(a, b, 0, 1)$, the mgf of X reduces to

$$M(t) = \frac{1}{\sqrt{\pi}}\sum_{j,k,m=0}^{\infty} w_{j,k}(a,b)\; c_{m,k}\; 2^{-t}\Gamma\left(m+\frac{k+1}{2}-t\right),$$

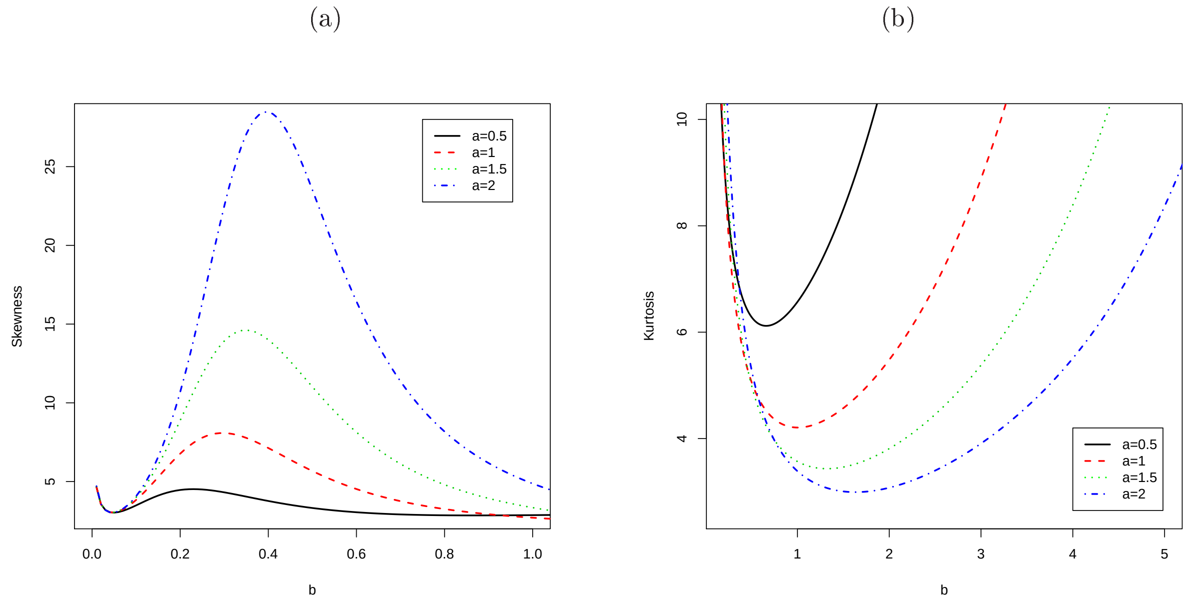

Figure 4: Skewness and kurtosis of the standard BMo distribution as a function of b for selected values of a.

where $w_{j,k}(a,b)$ and $c_{m,k}$ are defined in Sections 2 and 3.1, respectively.

Proof:

The mgf of the standard BMo distribution is

$$M(t) = \frac{1}{\sqrt{2\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) \int_{-\infty}^{\infty} \exp(tx) \exp\{-\frac{1}{2}[x + \exp(-x)]\} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^k dx.$$

Substituting $u = \exp(-x)/2$, we have

$$M(t) = \frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b)\ 2^{-t} \int_0^{\infty} u^{-\frac{1}{2}-t}\ e^{-u} \left[\gamma\left(\frac{1}{2}, u\right)\right]^k du. \tag{20}$$

Following similar steps of Theorem 1, $M(t)$ takes the form

$$M(t) = \frac{1}{\sqrt{\pi}} \sum_{j,k,m=0}^{\infty} w_{j,k}(a,b)\ c_{m,k}\ 2^{-t} \int_0^{\infty} u^{m+\frac{k+1}{2}-t-1}\ e^{-u} du.$$

By the definition of the gamma function, we obtain

$$M(t) = \frac{1}{\sqrt{\pi}} \sum_{j,k,m=0}^{\infty} w_{j,k}(a,b) c_{m,k}\ 2^{-t}\ \Gamma\left(m + \frac{k+1}{2} - t\right). \blacksquare$$

3.3 Means Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median - defined by

$$\delta_1(X) = \int_{-\infty}^{\infty} |\ x - \mu\ |\ dx \quad \text{and} \quad \delta_2(X) = \int_{-\infty}^{\infty} |\ x - M\ |\ dx,$$

respectively, where $\mu = \mathrm{E}(X)$ and $\mathrm{M} = Median(X)$ is the median. These measures can be expressed as

$$\delta_1(X) = 2\mu\mathrm{F}(\mu) - 2T(\mu) \text{ and } \delta_2(X) = \mu - 2T(M),$$

where $T(q) = \int_{-\infty}^{q} x f(x)dx$. Clearly, $F(\mu)$ and $F(M)$ are easily calculated from equation (6) and the median M satisfies the equation $\gamma(1/2, e^{-M}/2) = \sqrt{\pi}/2$. We have from (16)

$$T(q) = \frac{1}{\sqrt{2\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) \int_{-\infty}^{q} x \exp\{-[x + \exp(-x)]/2\} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^{k} dx.$$

We obtain by transforming $u = \exp(-x)/2$

$$T(q) = -\frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) \int_{\frac{1}{2}\exp(-q)}^{\infty} \log(2u)\ u^{-\frac{1}{2}} \exp(-u) \left[\gamma\left(\frac{1}{2}, u\right)\right]^{k} du.$$

Following similar steps from Theorem 1, we can write

$$\begin{aligned} T(q) &= -\frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) c_{m,k} \left\{\log(2)\Gamma\left(m + \frac{k+1}{2}, \frac{1}{2}\exp(-q)\right)\right. \\ &\quad \left. + \int_{\frac{1}{2}\exp(-q)}^{\infty} \log(u)\ u^{m+\frac{k+1}{2}-1} \exp(-u)\ du\right\}, \end{aligned}$$

where $\Gamma\left(m + \frac{k+1}{2}, \frac{\exp(-q)}{2}\right) = \int_{\frac{\exp(-q)}{2}}^{\infty} u^{m+\frac{k+1}{2}-1} \exp(-u)du$ is the complementary gamma function and $c_{m,k}$ is defined in Section 3.1. Then,

$$\begin{aligned} T(q) &= -\frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) c_{m,k} \left\{\log(2)\Gamma\left(m + \frac{k+1}{2}, \frac{\exp(-q)}{2}\right)\right. \\ &\quad \left. + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{\frac{\exp(-q)}{2}}^{\infty} u^{m+r+\frac{k+1}{2}-1} \log(u) du\right\}. \end{aligned}$$

Calculating the integral in the last equation by Maple, we obtain

$$\begin{aligned} T(q) &= -\frac{1}{\sqrt{\pi}} \sum_{j,k=0}^{\infty} w_{j,k}(a,b) c_{m,k} \left\{\log(2)\Gamma\left(m + \frac{k+1}{2}, \frac{1}{2}\exp(-q)\right)\right. \\ &\quad \left. + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left[\frac{\exp(-q)}{2}\right]^{\left(m+r+\frac{k+1}{2}\right)} \left[\left(m + r + \frac{k+1}{2}\right)\frac{q}{2} + 1\right] \left(m + r + \frac{k+1}{2}\right)^{-2}\right\}. \end{aligned} \quad (21)$$

The measures $\delta_1(X)$ and $\delta_2(X)$ are immediately calculated from (21).

3.4 Rényi Entropy

The entropy of a random variable is a measure of variation of the uncertainty. Entropy has been used in various situations in science and engineering, and numerous measures of entropy have been studied and compared in literature. For a BMo distribution (8), the Rényi entropy is defined by

$$\jmath_R(\xi) = \frac{1}{1-\xi} \log\left[I(\xi)\right],$$

where $I(\xi) = \int f^{\xi}(x)dx$, $\xi > 0$ and $\xi \neq 1$. From the BMo density function (8), we have

$$\begin{aligned} I(\xi) &= \frac{(\sqrt{2\pi})^{-\xi}}{[B(a,b)]^{\xi}\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}} \times \\ &\quad \int_{-\infty}^{\infty} e^{\{-\frac{\xi}{2}[x+\exp(-x)]\}} \left\{1 - \frac{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]}{\Gamma\left(\frac{1}{2}\right)}\right\}^{\xi(a-1)} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^{\xi(b-1)} dx. \end{aligned} \quad (22)$$

Using the series expansion (10) in (22), we obtain

$$\begin{aligned} I(\xi) &= \frac{(\sqrt{2\pi})^{-\xi}}{[B(a,b)]^{\xi}\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}} \sum_{j_1=0}^{\infty} \frac{(-1)^{j_1}\Gamma(\xi(a-1)+1)}{\Gamma(\xi(a-1)+1-j_1)j_1!\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}} \\ &\times \int_{-\infty}^{\infty} \exp\{-\frac{\xi}{2}[x+\exp(-x)]\}\left\{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]\right\}^{\xi(b-1)+j_1}. \end{aligned}$$

Using again this expansion and then applying the binomial expansion, we obtain

$$\begin{aligned} I(\xi) &= \frac{(\sqrt{2\pi})^{-\xi}}{[B(a,b)]^{\xi}\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}} \sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1} \frac{(-1)^{j_1+k_1+r_1}\Gamma(\xi(a-1)+1)\Gamma(\xi(b-1)+j_1+1)\binom{k_1}{r_1}}{\Gamma(\xi(a-1)+1-j_1)\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1} j_1!k_1!} \\ &\times \int_{-\infty}^{\infty} \exp\{-\frac{\xi}{2}[x+\exp(-x)]\}\left\{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]\right\}^{r_1}. \end{aligned}$$

Setting $u=\exp(-x)/2$, $I(\xi)$ reduces to

$$\begin{aligned} I(\xi) &= \frac{(\sqrt{\pi})^{-\xi}}{[B(a,b)]^{\xi}\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}} \sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1} \frac{(-1)^{j_1+k_1+r_1}\Gamma(\xi(a-1)+1)\Gamma(\xi(b-1)+j_1+1)\binom{k_1}{r_1}}{\Gamma(\xi(a-1)+1-j_1)\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1} j_1!k_1!} \\ &\times \int_{0}^{\infty} u^{\frac{\xi}{2}-1}\exp(-\xi u)\left[\gamma\left(\frac{1}{2},u\right)\right]^{r_1} du. \end{aligned}$$

Following similar developments in Theorem 1, we have

$$\begin{aligned} I(\xi) &= \frac{(\sqrt{\pi})^{-\xi}}{[B(a,b)]^{\xi}\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}} \\ &\times \sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1}\sum_{m_1}^{\infty} \frac{(-1)^{j_1+k_1+r_1}\Gamma(\xi(a-1)+1)\Gamma(\xi(b-1)+j_1+1)\binom{k_1}{r_1}c_{m_1,r_1}}{\Gamma(\xi(a-1)+1-j_1)\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1} j_1!k_1!} \\ &\times \int_{0}^{\infty} u^{m_1+\frac{\xi+r_1}{2}-1}\exp(-\xi u)du, \end{aligned} \tag{23}$$

where c_{m_1,r_1} is defined in Section 3.1. The integral in equation (23) can be easily calculated from the result given by Prudnikov *et al.* (1986, Vol.1, Section 2.3.3, integral 1). Hence,

$$\begin{aligned} I(\xi) &= \frac{(\sqrt{\pi})^{-\xi}}{[B(a,b)]^{\xi}\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}} \\ &\times \sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1}\sum_{m_1}^{\infty} \frac{(-1)^{j_1+k_1+r_1}\Gamma(\xi(a-1)+1)\Gamma(\xi(b-1)+j_1+1)\binom{k_1}{r_1}c_{m_1,r_1}}{\Gamma(\xi(a-1)+1-j_1)\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1} j_1!k_1!} \\ &\times \xi^{\left(m_1+\frac{\xi+r_1}{2}\right)}\Gamma\left(m_1+\frac{\xi+r_1}{2}\right). \end{aligned}$$

Finally, the Rényi entropy can be rewritten as

$$\begin{aligned} \jmath_R(\xi) &= (1-\xi)^{-1}\left\{-\xi\log(\sqrt{\pi})-\xi\log[B(a,b)]-\xi(b-1)\log\left[\Gamma\left(\frac{1}{2}\right)\right]\right. \\ &+ \log\left\{\sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1}\sum_{m_1}^{\infty} \frac{(-1)^{j_1+k_1+r_1}\Gamma(\xi(a-1)+1)\Gamma(\xi(b-1)+j_1+1)\binom{k_1}{r_1}c_{m_1,r_1}}{\Gamma(\xi(a-1)+1-j_1)\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1} j_1!k_1!}\right\} \\ &+ \left.\left(m_1+\frac{\xi+r_1}{2}\right)\log(\xi)+\log\left[\Gamma\left(m_1+\frac{\xi+r_1}{2}\right)\right]\right\}. \end{aligned} \tag{24}$$

4 Expansions for the order statistics

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, in a random sample of size n from the BMo distribution. It is well-known that

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} \left\{1 - F(x)\right\}^{n-i},$$

for $i = 1, \ldots, n$. Inserting (1) and (2) in the above equation, $f_{i:n}(x)$ can be written as

$$f_{i:n}(x) = \frac{n!g(x)}{(i-1)!(n-i)!B(a,b)} G(x)^{a-1}[1-G(x)]^{b-1}\left[I_{G(x)}(a,b)\right]^{i-1}\left[1 - I_{G(x)}(a,b)\right]^{n-i}.$$

Substituting (6) and (7) in the last equation, the density $f_{i:n}(x)$ for $b > 0$ real non-integer becomes

$$\begin{aligned} f_{i:n}(x) &= \sum_{k=0}^{n-i} \frac{(-1)^k \binom{n-i}{k} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}[x + \exp(-x)]\} \left\{1 - \frac{\gamma\{\frac{1}{2}, \frac{1}{2}\exp(-x)\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)-1}}{[\Gamma(b)^{i+k-1}]^{-1} B(a,b)^{i+k} B(i, n-1+i) \Gamma(\frac{1}{2})^{(b-1)} \left\{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp(-x)\right\}\right\}^{-(b-1)}} \\ &\times \left[\sum_{j=0}^{\infty} \frac{(-1)^j \left\{1 - \frac{\gamma\{\frac{1}{2}, \frac{1}{2}\exp(-x)\}}{\Gamma(\frac{1}{2})}\right\}^j}{\Gamma(b-j) j! (a+j)}\right]^{i+k-1}. \end{aligned} \quad (25)$$

We define

$$A = \sum_{j=0}^{\infty} \frac{(-1)^j \left\{1 - \frac{\gamma\{\frac{1}{2}, \frac{1}{2}\exp(-x)\}}{\Gamma(\frac{1}{2})}\right\}^j}{\Gamma(b-j) j! (a+j)}.$$

Setting $u = \exp(-x)/2$ and using the series expansion

$$1 - \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = x^{\alpha-1} \exp(-x) \sum_{m=0}^{\infty} \frac{x^{-m}}{\Gamma(\alpha - m)},$$

the quantity A becomes

$$A = \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j \; u^{-\frac{j}{2}} \; \exp(-uj) \left[\sum_{m=0}^{\infty} \frac{u^{-m}}{\Gamma(\frac{1}{2} - m)}\right]^j}{\Gamma(b-j) j! (a+j)} \right\}.$$

Hence,

$$A = \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j u^{-\frac{3j}{2}} \exp(-uj) \left[\sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{u^{-(m_1 + \cdots + m_j)}}{\Gamma(\frac{1}{2} - m_1) \ldots \Gamma(\frac{1}{2} - m_j)}\right]}{\Gamma(b-j) j! (a+j)} u^j \right\}. \quad (26)$$

We use the identity $(\sum_{k=0}^{\infty} a_k x^k)^n = \sum_{k=0}^{\infty} c_{k,n} x^k$ (see Gradshteyn and Ryzhik, 2000), where a_k now comes by identifying (26) with the corresponding quantity which is elevated to the power $i+k-1$ in equation (25). We have

$$a_k = \frac{(-1)^k\ u^{-\frac{3k}{2}}\ \exp(-uk)\left[\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\frac{u^{-(m_1+\cdots+m_k)}}{\Gamma(\frac{1}{2}-m_1)\ldots\Gamma(\frac{1}{2}-m_k)}\right]}{\Gamma(b-k)k!(a+k)},$$

$$c_{0,n} = a_0^n \qquad \text{and} \qquad c_{k,n} = (ka_0)^{-1}\sum_{l=1}^{k}(nl-k+l)a_l c_{k-l,n}$$

for $k = 1, 2, \ldots$ Thus, after some algebra, we obtain

$$f_{i:n}(x) = \sum_{k=0}^{n-i}\sum_{j=0}^{\infty}\frac{(-1)^k\binom{n-i}{k}\Gamma(b)^{i+k-1}B[a(i+k)+j,b]d_{i,j,k}}{B(a,b)^{i+k}B(i,n-1+i)}f_{i,j,k}(x), \tag{27}$$

where

$$\begin{aligned} f_{i,j,k}(x) \quad = \quad & \frac{\exp\{-\frac{1}{2}[x+\exp(-x)]\}}{\sqrt{2\pi}B[a(i+k)+j,b]\Gamma\left(\frac{1}{2}\right)^{b-1}}\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)+j-1} \times \\ & \left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}\right\}^{b-1} \end{aligned}$$

denotes the density of the BMo$(a(i+k)+j, b, 0, 1)$ distribution and the constants $d_{i,j,k}$ can be obtained recursively from the following equations

$$d_{i,0,k} = \left\{\frac{1}{a\Gamma(b)}\right\}^{i+k-1} \qquad \text{and} \qquad d_{i,j,k} = \frac{a\Gamma(b)}{j}\sum_{l=1}^{j}\frac{(-1)^l\{l(i+k)-j\}}{\Gamma(b-l)(a+l)l!}c_{j-l,i+k-1},\ \ j \geq 1.$$

The density function of the BMo order statistics is then an infinite mixture of BMo density functions. Hence, the ordinary and central moments of the order statistics can be calculated directly from the corresponding quantities of the BMo distribution given in Section 3. For $b > 0$ integer, expansion (27) holds but the sum in j stops at $(b-1)(k+i-1)$.

An alternative expansion for the density of the order statistics follow from the identity $(\sum_{i=0}^{\infty} a_i)^k = \sum_{m_1,\ldots,m_k=0}^{\infty} a_{m_1}\ldots a_{m_k}$ for k a positive integer. Using this identity and equation (25), for $b > 0$ real non-integer, it turns out that

$$\begin{aligned} f_{i:n}(x) \quad = \quad & \sum_{k=0}^{n-i}\sum_{m_1=0}^{\infty}\cdots\sum_{m_{i+k-1}=0}^{\infty}\frac{(-1)^k\binom{n-i}{k}\Gamma(b)^{i+k-1}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}[x+\exp(-x)]\}\left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}\right\}^{(b-1)}}{B(a,b)^{i+k}B(i,n-1+i)\Gamma(\frac{1}{2})^{(b-1)}} \\ & \times\frac{(-1)^{k+\sum_{j=1}^{i+k-1}m_j}\binom{n-i}{k}\Gamma(b)^{i+k-1}\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)+\sum_{j=1}^{i+k-1}m_j}}{\prod_{j=1}^{i+k-1}\Gamma(b-m_j)m_j!(a+m_j)}. \end{aligned}$$

Hence,

$$f_{i:n}(x) = \sum_{k=0}^{n-i}\sum_{m_1=0}^{\infty}\cdots\sum_{m_{i+k-1}=0}^{\infty}\delta_{i,k}f_{i,k}(x), \tag{28}$$

where

$$f_{i,k}(x) = \frac{\exp\{-\frac{1}{2}[x+\exp(-x)]\}}{\sqrt{2\pi}B[a(i+k)+\sum_{j=1}^{i+k-1} m_j, b]\Gamma\left(\frac{1}{2}\right)^{b-1}} \times$$
$$\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)+\sum_{j=1}^{i+k-1} m_j - 1} \left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}\right\}^{b-1}$$

denotes the density of the $\mathrm{BMo}(a(i+k)+\sum_{j=1}^{i+k-1} m_j, b, 0, 1)$ distribution and

$$\delta_{i,k} = \frac{(-1)^{k+\sum_{j=1}^{i+k-1} m_j}\binom{n-i}{k}B(a(i+k)+\sum_{j=1}^{i+k-1} m_j, b)\Gamma(b)^{i+k-1}}{B(a,b)^{i+k}B(i,n-i+1)\prod_{j=1}^{i+k-1}\Gamma(b-m_j)m_j!(a+m_j)}.$$

The constants $\delta_{i,k}$ are easily obtained given i, n, k and a sequence of indices $m_1, \ldots, m_{i+k-1}$. The sums in (28) extend over all $(i+k)$-tuples $(k, m_1, \ldots, m_{i+k-1})$ of non-negative integers and can be implementable on a computer. If $b > 0$ is an integer, equation (28) holds but the indices $m_1, \ldots, m_{i+k-1}$ vary from zero to $b-1$. Expansion (27) is much simpler to be calculated numerically in applications and the corresponding CPU times are usually smaller than those using (28).

The sth moment of $X_{i:n}$ for $b > 0$ real non-integer comes from (27)

$$\mathrm{E}(X_{i:n}^s) = \sum_{k=0}^{n-i}\sum_{j=0}^{\infty} \frac{(-1)^k\binom{n-i}{k}\Gamma(b)^{i+k-1}B(a(i+k)+j,b)d_{i,j,k}}{B(a,b)^{i+k}B(i,n-i+1)} E(X_{i,j,k}^s), \tag{29}$$

where $X_{i,j,k} \sim \mathrm{BMo}(a(i+k)+j, b, 0, 1)$ and the constants $d_{i,j,k}$ were defined before. If b is an integer, the sum in j stops at $b-1$.

From equation (28), we can obtain an alternative expression for the moments of the order statistics valid for $b > 0$ real non-integer

$$\mathrm{E}(X_{i:n}^s) = \sum_{k=0}^{n-i}\sum_{m_1=0}^{\infty}\cdots\sum_{m_{i+k-1}=0}^{\infty} \delta_{i,k} E(X_{i,k}^s), \tag{30}$$

where $X_{i,k} \sim \mathrm{BMo}(a(i+k)+\sum_{j=1}^{i+k-1} m_j, b, 0, 1)$. For $b > 0$ integer, the indices $m_1, \ldots, m_{i+k-1}$ stop at $b-1$.

We therefore offer two equations (29) and (30) for the moments of the BMo order statistics which are the main results of this section.

Based upon the moments given in equations (29) and (30), we can easily derive expansions for the L-moments of the BMo distribution as infinite weighted linear combinations of the means of suitable BMo distributions. The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. They are linear functions of expected order statistics defined by (Hosking, 1990)

$$\lambda_{r+1} = r(r+1)^{-1}\sum_{k=0}^{r}\frac{(-1)^k}{k}\mathrm{E}(X_{r+1-k:r+1}),\ r = 0, 1, \ldots$$

The first four L-moments are: $\lambda_1 = \mathrm{E}(X_{1:1})$, $\lambda_2 = \frac{1}{2}\mathrm{E}(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}\mathrm{E}(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}\mathrm{E}(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$.

5 Estimation and Inference

The parameters of the BMo distribution are estimated by the method of maximum likelihood. If X follows the BMo distribution with vector of parameters $\boldsymbol{\lambda} = (a, b, \mu, \sigma)^T$, the log-likelihood for the model parameters from a single observation x of X is given by

$$\begin{aligned}\ell(\boldsymbol{\lambda}) &= \log\left(\frac{1}{\sqrt{2\pi}}\right) - \log(\sigma) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right) - \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right] - \log\left[B(a,b)\right] \\ &- (b-1)\log\left[\Gamma\left(\frac{1}{2}\right)\right] + (a-1)\log\left\{1 - \frac{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma(\frac{1}{2})}\right\} \\ &+ (b-1)\log\left\{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\right\}, \qquad -\infty < x < \infty.\end{aligned}$$

The components of the unit score vector $\mathbf{U} = \left(\frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\mu}, \frac{\partial\ell}{\partial\sigma}\right)^T$ are

$$\frac{\partial\ell}{\partial a} = \log\left\{1 - \frac{\gamma\left[\frac{1}{2}, \frac{1}{2}\exp(-z)\right]}{\Gamma(\frac{1}{2})}\right\} - \psi(a) + \psi(a+b),$$

$$\frac{\partial\ell}{\partial b} = \log\left\{\gamma\left[\frac{1}{2}, \frac{1}{2}\exp(-z)\right]\right\} - \log\left[\Gamma\left(\frac{1}{2}\right)\right] - \psi(b) + \psi(a+b),$$

$$\begin{aligned}\frac{\partial\ell}{\partial\mu} &= \frac{1}{2\sigma} - \frac{1}{2\sigma}\exp(-z) + (a-1)\left\{\frac{\frac{1}{2\sigma}\Gamma\left(\frac{1}{2}\right)(\sqrt{2\pi}\sigma)^{-1}\exp\left[-z-\exp(-z)\right]\left[1+\exp(-z)\right]}{\Gamma\left(\frac{1}{2}\right) - \gamma\left[\frac{1}{2}, \frac{1}{2}\exp(-z)\right]}\right\} \\ &+ (b-1)\left\{\frac{\frac{\sqrt{2}}{2\sigma}\exp\left[-\frac{1}{2}\exp(-z)\right]}{\gamma\left[\frac{1}{2}, \frac{1}{2}\exp(-z)\right]}\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{z}{2\sigma} - \frac{z}{2\sigma}\exp(-z) + (b-1)\left\{\frac{\frac{z\sqrt{2}}{2\sigma}\exp\left\{-\frac{1}{2}\left[z+\exp(-z)\right]\right\}}{\gamma\left[\frac{1}{2}, \frac{1}{2}\exp(-z)\right]}\right\} \\ &+(a-1)\left\{\frac{\Gamma\left(\frac{1}{2}\right)(\sqrt{2\pi}\sigma^2)^{-1}\exp\left\{-\frac{1}{2}\left[z+\exp(-z)\right]\right\}\left[\frac{z}{2} + z\exp(-z) - 1\right]}{\Gamma\left(\frac{1}{2}\right) - \gamma\left[\frac{1}{2}, \frac{1}{2}\exp(-z)\right]}\right\},\end{aligned}$$

where $z = (x-\mu)/\sigma$ and $\psi(.)$ is the digamma function.

For a random sample $\mathbf{x} = (x_1, \ldots, x_n)^T$ of size n from X, the total log-likelihood is $\ell_n = \ell_n(\boldsymbol{\lambda}) = \sum_{i=1}^{n} \ell^{(i)}(\boldsymbol{\lambda})$, where $\ell^{(i)}(\boldsymbol{\lambda})$ is the log-likelihood for the ith observation ($i = 1, \ldots, n$). The total score function is $\mathbf{U}_n = \sum_{i=1}^{n} \mathbf{U}^{(i)}$, where $\mathbf{U}^{(i)}$ has the form given before for $i = 1, \ldots, n$. The maximum likelihood estimate (MLE) $\hat{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ is the solution of the nonlinear system of equations $\mathbf{U}_n = \mathbf{0}$.

For interval estimation and tests of hypotheses on the parameters in $\boldsymbol{\lambda}$, we require the 4×4 unit expected information matrix

$$\mathbf{K} = \mathbf{K}(\boldsymbol{\lambda}) = \begin{pmatrix} \kappa_{aa} & \kappa_{a,b} & \kappa_{a,\mu} & \kappa_{a,\sigma} \\ . & \kappa_{b,b} & \kappa_{b,\mu} & \kappa_{b,\sigma} \\ . & . & \kappa_{\mu,\mu} & \kappa_{\mu,\sigma} \\ . & . & . & \kappa_{\sigma,\sigma} \end{pmatrix},$$

whose elements are given in Appendix A.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda})$ is $\mathrm{N}_4(0, \mathbf{K}(\boldsymbol{\lambda})^{-1})$. The estimated asymptotic multivariate normal $\mathrm{N}_4(0, n^{-1}\mathbf{K}(\hat{\boldsymbol{\lambda}})^{-1})$ distribution of $\hat{\boldsymbol{\lambda}}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An asymptotic confidence interval with significance level γ for each parameter λ_r is given by

$$\mathrm{ACI}(\lambda_r, 100(1-\gamma)\%) = (\hat{\lambda}_r - z_{\gamma/2}\sqrt{\hat{\kappa}^{\lambda_r,\lambda_r}}, \hat{\lambda}_r + z_{\gamma/2}\sqrt{\hat{\kappa}^{\lambda_r,\lambda_r}}),$$

where $\hat{\kappa}^{\lambda_r,\lambda_r}$ is the rth diagonal element of $n^{-1}\mathbf{K}(\boldsymbol{\lambda})^{-1}$ estimated at $\hat{\boldsymbol{\lambda}}$, for $r = 1, \cdots, 4$, and $z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for testing goodness-of-fit of the BMo distribution and for comparing this distribution with some of its special sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics for testing some sub-models of the BMo distribution. For example, we may use the LR statistic to check if the fit using the BMo distribution is statistically "superior" to a fit using the Moyal distribution for a given data set. In any case, considering the partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^T, \boldsymbol{\lambda}_2^T)^T$, tests of hypotheses of the type $H_0 : \boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_1^{(0)}$ versus $H_A : \boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_1^{(0)}$ can be performed via the LR statistic $w = 2\{\ell(\hat{\boldsymbol{\lambda}}) - \ell(\tilde{\boldsymbol{\lambda}})\}$, where $\hat{\boldsymbol{\lambda}}$ and $\tilde{\boldsymbol{\lambda}}$ are the MLEs of $\boldsymbol{\lambda}$ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector $\boldsymbol{\lambda}_1$ of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution. From the score vector and the information matrix given before, we can also construct the score and Wald statistics which are asymptotically equivalent to the LR statistic.

6 Applications

In this section, we give two applications using well-known data sets in order to demonstrate the flexibility and applicability of the proposed model over other parametric models.

6.1 Airborne data

The first data set represents repair times (in h) for an airborne communication transceiver. They were first analyzed by Von Alven (1964) who fitted the two-parameter log-normal distribution. Chhikara and Folks (1977) fitted a two-parameter inverse Gaussian distribution. Recently, the data were reanalyzed by Koutrouvelis *et al.* (2005) using the inverse Gaussian distribution with three parameters. Table 1 lists the MLEs (and the corresponding standard errors in parentheses) of the model parameters and the values of the following statistics for some models: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion). The computations were done using the subroutine NLMIXED in SAS. These results indicate that the beta Moyal model has the lowest AIC, BIC and CAIC values among the values of the fitted models, and therefore it could be chosen as the best model.

In order to assess if the model is appropriate, Figure 5 gives the histogram of the data and plots of the fitted BMo, Moyal, beta normal and skew normal distributions. The BMo distribution provides a good fit for these data.

6.2 The wheaton river data

As a second example, we consider the data set presented by Akinsete *et al.* (2008). The data are the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place. These data were analyzed by Choulakian and Stephens (2001). Table 2 gives the MLEs of the model parameters. In any case, since the values of the three statistics are smaller for the BMo distribution compared to those values of the Moyal, beta normal and skew-normal distributions, the new distribution seems to be a very competitive model for lifetime data analysis. The histogram of the data and the plots of the fitted BMo,

Table 1: MLEs of the model parameters for the airborne data, the corresponding SEs (given in parentheses) and the statistics AIC, CAIC and BIC.

Model	a	b	μ	σ	AIC	CAIC	BIC
BMo	0.1108	0.3356	3.8656	0.9961	213.3	214.3	220.5
	(0.0181)	(0.0790)	(0.0012)	(0.00008)			
Moyal	1	1	1.5596	1.3318	227.3	227.6	230.9
	-	-	(0.2925)	(0.1835)			
Beta normal	0.4328	0.3082	1.6485	2.7719	285.2	286.2	292.4
	(0.1262)	(0.0727)	(1.9549)	(0.2513)			
		λ	μ	σ			
Skew-normal	-	0.9008	0.1877	4.8804	275.3	275.9	280.7
	-	(0.7689)	(1.4426)	(0.5858)			

Table 2: MLEs of the model parameters for the wheaton river data, the corresponding SEs (given in parentheses) and the statistics AIC, CAIC and BIC.

Model	a	b	μ	σ	AIC	CAIC	BIC
BMo	0.2693	0.2612	5.5967	2.0295	257.5	258.1	266.6
	(0.0370)	(0.0479)	(0.4095)	(0.2230)			
Moyal	1	1	5.4092	4.8127	271.6	271.8	276.1
	-	-	(0.8569)	(0.5126)			
Beta normal	141.14	134.0	6.9376	161.35	572.6	573.2	581.7
	(0.4302)	(0.3954)	(1.8309)	(13.4563)			
		λ	μ	σ			
Skew-normal	-	-0.0026	12.2279	12.2125	570.7	571.0	577.5
	-	(4.3820)	(2.7227)	(1.0214)			

Moyal, beta normal and skew-normal distributions are given in Figure 6. These plots show some evidence that the BMo distribution seems superior to the other distributions in terms of model fit.

7 Conclusions

In this article, we propose a new model so-called the beta Moyal (BMo) distribution to extend the Moyal distribution in the analysis of skew data with real support. An obvious reason for generalizing a "standard distribution" is because the generalized form provides greater flexibility in modeling real data. We provide a mathematical treatment of the new distribution including expansions for its distribution and density functions. We derive expansions for the moments, the moment generating function, the mean deviations and the moments of order statistics. The estimation of parameters is approached by the method of maximum likelihood and the information matrix is derived. We consider the likelihood ratio (LR) statistic to compare the model with its baseline model. Two applications of the BMo distribution to real data show that the new distribution can be used quite effectively to provide better fits than the beta normal, Moyal and skew-normal distributions.

Acknowledgments

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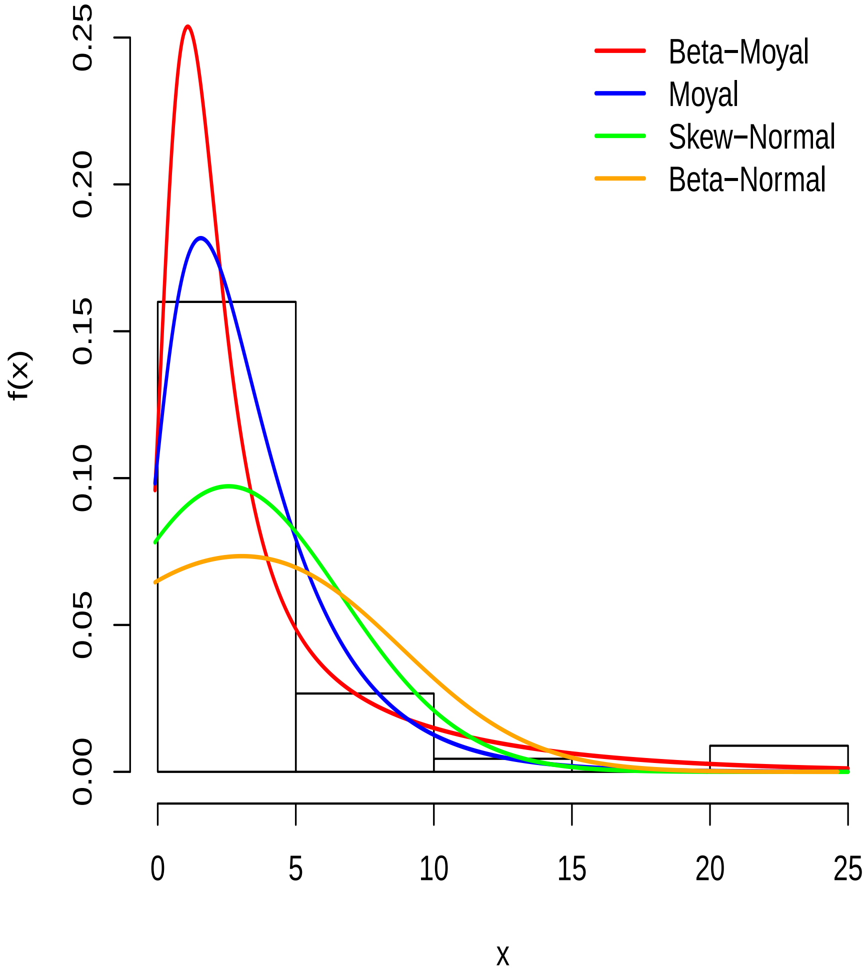

Figure 5: Fitted densities of the BMo, Moyal, beta normal and skew-normal models for the airborne data.

Appendix A

The elements of the 4×4 unit expected information matrix are given by

$$\begin{aligned}
\kappa_{\sigma,\sigma} \;=\; & -\frac{1}{\sigma} - \frac{1}{2\sigma^2}\left[T_{0,0,2,0,1,0,0} - T_{0,0,2,0,2,0,0}\right] - \frac{\sqrt{2}(a-1)}{\sigma^3\sqrt{\pi}}\left\{\frac{2}{\sqrt{\pi}}\left[(1+\log(2))\, T_{1,0,1,1,0,0,0}\right.\right. \\
& + \left.\log(4)\, T_{1,0,2,1,0,0,0} - 4\, T_{1,0,3,1,0,1,0} + T_{1,0,1,1,0,1,0}\right] + \frac{1}{2\sqrt{\pi}}\left[\frac{1}{2}\, T_{1,0,1,1,2,0,0}\right. \\
& + T_{1,0,3,1,2,0,0} - 4\, T_{1,0,5,1,2,0,0} - 2\, T_{1,0,3,1,1,0,0} - 4\, T_{1,0,3,1,1,0,1} + 2\, T_{1,0,2,2,2,0,0} \\
& \left.\left. + 8\, T_{1,0,4,2,2,0,0}\right] + \frac{2}{\pi}\, T_{2,0,2,2,1,0,0}\right\} \;-\sqrt{2}(b-1)\left\{\frac{1}{\sqrt{\pi}}\, T_{0,1,1,1,1,0,0}\right. \\
& \left. + \frac{1}{4\sqrt{\pi}\sigma^2}\left[\sqrt{2}\, T_{0,1,1,1,2,0,0} - 2\, T_{0,1,3,1,2,0,0}\right] - \frac{\sqrt{2}}{2\pi\sigma^2}\, T_{0,2,2,2,2,0,0}\right\},
\end{aligned}$$

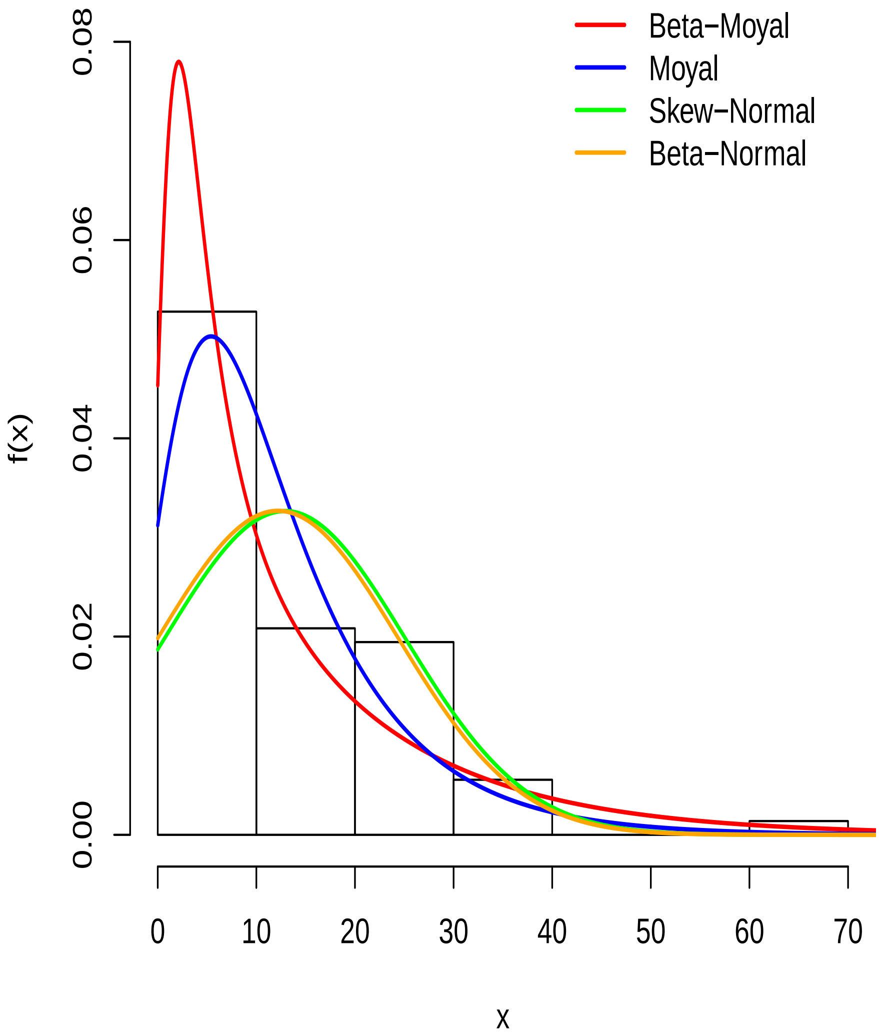

Figure 6: Estimated densities of the BMo, Moyal, beta normal and skew-normal models for the wheaton river data.

$$\kappa_{\mu,\sigma} = \frac{1}{2\sigma^2} - \frac{1}{\sigma^2}\left[T_{0,0,2,0,0,0,0} + T_{0,0,2,0,1,0,0}\right] - \frac{(a-1)}{\sigma^3}\left\{\frac{1}{\sqrt{\pi}}\left[\frac{1}{4}\,T_{1,0,1,1,1,0,0} + \frac{3}{2}\,T_{1,0,3,1,1,0,0}\right.\right.$$
$$\left.\left. + 3\,T_{1,0,3,1,0,0,0} - T_{1,0,1,1,0,0,0}\right] + \frac{1}{2\pi}\left[T_{2,0,2,2,1,0,0} + 4\,T_{2,0,4,2,1,0,0} + 2\,T_{2,0,2,2,0,0,0}\right]\right\}$$
$$+\frac{(b-1)}{\sigma^2}\left\{\frac{1}{\sqrt{\pi}}\,T_{0,1,1,1,0,0,0} - \frac{1}{\pi}\,T_{0,2,2,2,1,0,0} + \frac{1}{2\sqrt{\pi}}\left[T_{0,1,1,1,1,0,0} - 2\,T_{0,1,3,1,1,0,0}\right]\right\},$$

$$\kappa_{b,\mu} = -\frac{1}{2\sigma}\sqrt{\frac{2}{\pi}}\ T_{0,1,0,1,0,0,0}, \quad \kappa_{b,\sigma} = \frac{1}{\sigma\sqrt{\pi}} T_{0,1,1,1,1,0,0}, \quad \kappa_{a,b} = -\psi^{'}(a+b),$$

$$\begin{aligned}\kappa_{\mu,\mu} \ &= \ \frac{1}{\sigma^2} T_{0,0,2,0,0,0,0} - \frac{(a-1)}{\sigma^3\sqrt{2}}\left\{\frac{2}{\sqrt{\pi}}\left[\left(1-\frac{1}{\sigma}\right)\ T_{1,0,4,2,0,0,0} + \frac{2}{\sigma}\ T_{1,0,8,2,0,0,0}\right] + \frac{\sqrt{2}}{2\pi}\left[T_{2,0,2,3,0,0,0}\right.\right.\\ &\left.\left. + \ 2\ T_{2,0,4,3,0,0,0}\right]\right\} + \frac{(b-1)\sqrt{2}}{4\sigma^2}\left[\frac{2}{\sqrt{\pi}}\ T_{0,1,2,1,0,0,0} + \frac{\sqrt{2}}{\pi}\ T_{0,2,0,2,0,0,0}\right],\end{aligned}$$

$$\kappa_{a,\sigma} = \frac{1}{\pi\sigma^2}\left[(1+\log(2))\ \ T_{1,0,1,1,0,0,0} + \log(4)\ T_{1,0,2,1,0,0,0} + 4\ T_{1,0,3,1,0,1,0} + T_{1,0,1,1,0,1,0}\right],$$

$$\kappa_{a,\mu} = -\frac{1}{\sigma^2\sqrt{2\pi}}\left[T_{1,0,2,2,0,0,0} + 2\ T_{1,0,4,2,0,0,0}\right], \kappa_{a,a} = \psi^{'}(a) - \psi^{'}(a+b),\ \kappa_{b,b} = \psi^{'}(b) - \psi^{'}(a+b).$$

Here, we assume that a random variable V has a beta distribution with parameters a and b and define the expected value

$$\begin{aligned}T_{i,j,k,l,m,n,p} \ &= \ \mathrm{E}\left\{V^{-i}\,(1-V)^{-j}\left[\mathrm{erf}^{-1}(1-V)\right]^{k}\exp\left\{-l\left[\mathrm{erf}^{-1}(1-V)\right]^{2}\right\}\left[\log\left\{2\left[\mathrm{erf}^{-1}(1-V)\right]^{2}\right\}\right]^{m}\right.\\ &\left.\times\left[\log\left\{\mathrm{erf}^{-1}(1-V)\right\}\right]^{n}\left[\exp\left\{2\left[\mathrm{erf}^{-1}(1-V)\right]^{2}\log\left\{2\left[\mathrm{erf}^{-1}(1-V)\right]^{2}\right\}\right\}\right]^{p}\right\}.\end{aligned}$$

These expected values can be determined numerically using MAPLE and MATHEMATICA for any a and b. For example, for $a = 2.5$ and $b = 3$, we easily calculate all T's in the information matrix: $T_{0,0,2,0,1,0,0} = 0.4598979, T_{0,0,2,0,2,0,0} = 0.1271101, T_{0,1,1,1,1,0,0} = -0.4559037, T_{0,1,1,1,2,0,0} = 1.800166, T_{0,1,3,1,2,0,0} = 0.1077819$ $, T_{0,2,2,2,2,0,0} = 0.9190356$, $T_{0,0,2,0,0,0,0} = 0.4948074$, $T_{0,1,1,1,0,0,0} = 0.6847316, T_{0,2,2,2,1,0,0} = 0.291509, T_{0,1,3,1,1,0,0} = -0.08361248, T_{0,1,0,1,0,0,0} = 1.745719$, $T_{0,1,2,1,0,0,0} = 0.4406842, T_{0,2,0,2,0,0,0} = 3.955121, T_{1,0,1,1,0,0,0} = 0.9004557, T_{1,0,1,1,0,1,0} =, -0.8180274, T_{1,0,1,1,2,0,0} = 0.3652951, T_{1,0,1,1,1,0,0} = -0.2425763, T_{1,0,2,1,0,0,0} = 0.39603, T_{1,0,2,2,2,0,0} =, 0.04201742, T_{1,0,2,2,0,0,0} = 0.01161524, T_{1,0,3,1,0,1,0} = -0.2319999, T_{1,0,3,1,2,0,0} = 0.2246422, T_{1,0,3,1,1,0,0} = -0.2028589, T_{1,0,3,1,1,0,1} = -0.1049516, T_{1,0,3,1,0,0,0} = 0.7749216, T_{1,0,5,1,2,0,0} = 0.004927196, T_{1,0,4,2,2,0,0} = 0.003980552, T_{1,0,4,2,0,0,0} = 0.09122807, T_{2,0,2,2,1,0,0} = 1.632044, T_{2,0,2,2,0,0,0} = 2.917779$, $T_{2,0,2,3,0,0,0} = 2.567143, T_{2,0,4,2,1,0,0} = -0.5019668, T_{2,0,4,3,0,0,0} = 2.570722, T_{1,0,8,2,0,0,0} = 0.08920794$.

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