On the Continuity of Minimum Stable Distributions

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Abstract

It is a well known result in extreme value theory that there are only three possible non-degenerate limiting distributions for the sequence $M_n = \min\{\xi_1, \ldots, \xi_n\}$, $n \geq 1$, where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables defined on the same probability space. The result, due to Fisher and Tippett (1928) and Gnedenko (1943), appears in many textbooks of probability theory and its proof relies on the solution of certain functional equations. It is of great instructional value, however, the direct derivation of many interesting properties of these limiting distributions, from the notion of minimum stability and in this article a proof of their necessary continuity is presented.

Key Words: minimum stable distributions.

1 Introduction

Extreme value theory deals, among other things, with the derivation of aproximate distributions for the maximum and the minimum values observed in simple random samples. The formal setting is that in which

$$M_n = \min\{\xi_1, \ldots, \xi_n\} \ , \ n \geq 1 \ ,$$

where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables defined on the same probability space, $(\Omega, \mathcal{F}, P)$, and one is interested in:

1. whether there are real constants, $\alpha_n > 0$ and β_n, $n \geq 1$, and a distribution function, G, such that
$$\lim_{n\to\infty} P\{\frac{M_n - \beta_n}{\alpha_n} \leq x\} = G(x) \ ,$$
for all x in the continuity set of G; and, if this is the case,

2. what properties does G have.

In this context we say that G a limiting distribution for $\{M_n\}_{n=1}^{\infty}$ and it is well known that if G is non-degenerate, it must have one of the following types:

1. $G_1(x) = 1 - e^{-x^{\lambda}}$, for $x \geq 0$, where $\lambda > 0$;
2. $G_2(x) = 1 - e^{-(-x)^{-\lambda}}$, for $x \leq 0$, where $\lambda > 0$; or
3. $G_3(x) = 1 - e^{-e^{x}}$, for $-\infty < x < +\infty$.

A demonstration of the above result, due to Fisher and Tippett (1928) and Gnedenko (1943), can be found in Barlow and Proschan (1975) and its proof relies on the solution of the following three functional equations:

1. $\overline{G}^n(\alpha_n x) = \overline{G}(x)$, with $G(x) = 1 - \overline{G}(x) = 0$ for $x \leq 0$;
2. $\overline{G}^n(\alpha_n x) = \overline{G}(x)$, with $G(x) = 1 - \overline{G}(x) = 1$ for $x \geq 0$;
3. $\overline{G}^n(x + \beta_n) = \overline{G}(x)$, for $x \in \mathbb{R}$,

derived from the analysis of the normalizing real constants, $\alpha_n > 0$ and β_n, $n \geq 1$, in following result

Theorem 1. *G is a limiting distribution for $\{M_n\}_{n=1}^{\infty}$ if and only if G is minimum stable in the sense that there are real constants $\alpha_n > 0$ and β_n, $n \geq 1$ such that*
$$\overline{G}^n(\alpha_n x + \beta_n) = \overline{G}(x) \ , \quad \text{for all } x \in \mathbb{R} \ .$$

Remarks.

1. The proof of Theorem 1 is elementary. For details, see Barlow and Proschan (1975, page 232, Theorem 2.1)

2. For details of the afore mentioned analysis of the normalizing constants see Barlow and Proschan (1975, pages 232-234, Lemmas 2.2 - 2.6)

3. The solution of the above functional equations require the notion of regular variation, introduced by Karamata (1930). For details, see Barlow and Proschan (1975, pages 234-236, Lemma 2.7, definition 2.8, Lemma 2.9 and Theorem 2.10)

In spite of the mathematical beauty of the above described develoment, from the educational point of view several interesting properties of non-degenerate minimum stable distributions functions can be directly obtained from first principles in mathematical analysis. In the next section, we provide a tricky proof of the continuity of these distributions which, to the authors knowledge, is new.

2 Main Result

Theorem 2. Every non-degenerate minimum-stable distribution function, G, is continuous.

proof. For any distribution function, G, its continuity on $\{G = 0\}$ is immediate for if $x \in \mathbb{R}$ is such that $G(x) = 0$ then $G(y) = 0$ for any $y < x$, which implies that $G(x - 0) = G(x)$ (Recall that G is always right continuous).

In view of the above remark, to establish the continuity of a non-degenerate minimum-stable distribution function, G, it suffices to prove that G is also left continuous on $\{G > 0\}$ or, equivalently, that $\overline{G}$ is left continuous on $\{\overline{G} < 1\}$.

Recall from Theorem 1 that a distribution function G is minimum stable if and only if, for each $n \in \mathbb{N}$, there are real constants $\alpha_n > 0$ and β_n such that

$$\overline{G}^n(\alpha_n x + \beta_n) = \overline{G}(x) \quad \text{for all} \quad x \in \mathbb{R} \ .$$

Alternatively, G is minimum stable if and only if, for each $n \in \mathbb{N}$, there are real constants $\alpha'_n > 0$ and β'_n such that

$$\overline{G}^n(x) = \overline{G}(\alpha'_n x + \beta'_n) \quad \text{for all} \quad x \in \mathbb{R} \ .$$

Let us now fix $x \in \{\overline{G} < 1\}$ and assume that

$$\overline{G}(x - 0) - \overline{G}(x) = \epsilon > 0 \ .$$

If for some $y < x$

$$1 > \overline{G}(y) \geq 1 - \frac{\epsilon}{2} \ ,$$

then

$$\overline{G}^n(y) - \overline{G}^{n+1}(y) = \overline{G}^n(y)[1 - \overline{G}(y)] \leq \frac{\epsilon}{2}$$

and consequently

$$\overline{G}(x) \leq \overline{G}^k(y) = \overline{G}(\alpha'_k y + \beta'_k) \leq \overline{G}(x - 0) \ ,$$

for some integer $k \in \mathbb{N}$, since $\lim_{n\to\infty} \overline{G}^n(y) = 0$. But this contradicts the fact that $\overline{G}$ is discontinuous at x since $\overline{G}$ cannot take values between $\overline{G}(x)$ and $\overline{G}(x - 0)$.

From the above observation, in order to prove that $\overline{G}$ is left continuous on $\{\overline{G} < 1\}$, it suffices to show that for any $x \in \{\overline{G} < 1\}$ and $\epsilon > 0$, there exists $y < x$ such that

$$1 > \overline{G}(y) \geq 1 - \frac{\epsilon}{2} \ .$$

For that, let

$$z = \sup\{y < x \ : \ \overline{G}(y) \geq 1 - \frac{\epsilon}{2}$$

and observe that

$$\overline{G}(z) = \overline{G}(z+0) \leq 1 - \frac{\epsilon}{2} < 1 \quad (1)$$

and

$$\overline{G}(z) = \overline{G}^2(\alpha_2 z + \beta_2) \geq \overline{G}^2(z) \quad (2) \ ,$$

since $\overline{G}(z) < 1$.

Consequently,

$$\alpha_2 z + \beta_2 < z \quad \text{(since the fact that } G \text{ non-degenerate excludes equality}$$

and

$$1 > \overline{G}(\alpha_2 z + \beta_2) \geq 1 - \frac{\epsilon}{2} \ ,$$

for if $\overline{G}(\alpha_2 z + \beta_2) = 1$ would lead to a contradiction between (1) and (2).

QED

3 References

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