The Log-generalized inverse Weibull Regression Model

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Abstract

The inverse Weibull distribution has the ability to model failure rates which are quite common in reliability and biological studies. A three-parameter generalized inverse Weibull distribution. A location-scale regression model based on the generalized inverse Weibull distribution is proposed as an alternative model for modeling data when the failure rate function is unimodal, decreasing or an increasing function.

Keywords: Censored data; Data analysis; Inverse Weibull; Maximum likelihood estimation; Moments; Weibull regression model.

1 Introduction

The inverse Weibull distribution has received some attention in the literature. Keller and Kamath (1982) study the shapes of the density and failure rate functions for the basic inverse model.

Let T be the cumulative density function (cdf) of a random variable following the inverse Weibull distribution. Then, the cdf of T is given by

$$F(t) = \exp\left[-\left(\frac{\alpha}{t}\right)^{\beta}\right], \quad t > 0, \tag{1}$$

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where $\alpha, \beta > 0$. The corresponding probability density function (pdf) is given by

$$f(t) = \beta\alpha^{\beta}t^{-(\beta+1)}\exp\left[-\left(\frac{\alpha}{t}\right)^{\beta}\right]. \tag{2}$$

2 A generalized inverse Weibull distribution

Let $F_L(t;\alpha,\beta)$ be the cdf of the inverse Weibull distribution discussed by Drapella (1993), Mudholkar and Kollia (1994) and Jiang et al. (1999), among others. The cdf of the generalized inverse Weibull (GIW) distribution can be defined by elevating $F_L(t;\alpha,\beta)$ to the power of γ, namely $F(t) = F_L(t;\alpha,\beta)^{\gamma} = \exp\big[-\gamma\big(\frac{\alpha}{t}\big)^{\beta}\big]$. Hence, the density function of the GIW distribution with three parameters $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ is defined by

$$f(t) = \gamma\beta\alpha^{\beta}t^{-(\beta+1)}\exp\left[-\gamma\left(\frac{\alpha}{t}\right)^{\beta}\right], \quad t > 0. \tag{3}$$

We can easily prove that (3) is a density function by considering the substitution $u = -\gamma\alpha^{\beta}t^{-\beta}$. The inverse Weibull distribution is a special case of (3) when $\gamma = 1$. If T is a random variable with density (3), we write $T \sim$ GIW(α,β,γ).

3 A Log-generalized inverse Weibull Distribution

Let T be a random variable following the GIW density function (3). We define the random variable $Y = \log(T)$ having a log-generalized inverse Weibull (LGIW) distribution, parameterized in terms of $\beta = 1/\sigma$ and $\alpha = \exp(\mu)$, by the density function

$$f(y;\gamma,\sigma,\mu) = \frac{\gamma}{\sigma}\exp\left\{-\left(\frac{y-\mu}{\sigma}\right) - \gamma\exp\left[-\left(\frac{y-\mu}{\sigma}\right)\right]\right\}, \quad -\infty < y < \infty \tag{4}$$

where $\gamma > 0$, $\sigma > 0$ and $-\infty < \mu < \infty$. The corresponding survival function reduces to

$$S(y) = 1 - \exp\left\{-\gamma\exp\left[-\left(\frac{y-\mu}{\sigma}\right)\right]\right\}.$$

We define the random variable $Z = (Y-\mu)/\sigma$ with density function

$$f(z) = \gamma\exp\big\{-z - \gamma\exp(z)\big\}, \quad -\infty < z < \infty \quad \text{and} \quad \gamma > 0. \tag{5}$$

When $\gamma = 1$, we obtain the inverse extreme value distribution as a special case.

4 A Log-generalized inverse Weibull Regression Model

In many practical applications, the lifetimes are affected by explanatory variables such as the cholesterol level, blood pressure and many others. Let $\mathbf{x}_i = (x_{i1}, \ldots, x_{ip})^T$ be the explanatory variable vector associated with the ith response variable y_i for $i = 1, \ldots, n$. Based on the LGIW density, we construct a linear regression model linking the response variable y_i and the explanatory variable vector $\mathbf{x}_i$ as follows

$$y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \sigma z_i, \ i = 1, \ldots, n, \tag{6}$$

where the random error z_i has the distribution (5), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$, $\sigma > 0$ and $\gamma > 0$ are unknown parameters and $\mathbf{x}_i$ is the explanatory variable vector modeling the location parameter μ_i. Hence, the location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ of the LGIW regression model can be expressed as a linear model $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^T$ is a known model matrix. From now on, the log-inverse Weibull (LIW) (or inverse extreme value) regression model is defined from (6) by taking $\gamma = 1$.

Consider a sample $(y_1, \mathbf{x}_1), \ldots, (y_n, \mathbf{x}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(t_i), \log(c_i)\}$. We assume non-informative censoring and that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is log-lifetime or log-censoring, respectively. The total log-likelihood function for the model parameters $\boldsymbol{\theta} = (\gamma, \sigma, \boldsymbol{\beta}^T)^T$ follows from (5) and (6) as

$$\begin{aligned} l(\boldsymbol{\theta}) &= r\big[\log(\gamma) - \log(\sigma)\big] - \sum_{i \in F}\left(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}}{\sigma}\right) - \gamma \sum_{i \in F} \exp\left\{-\left(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}}{\sigma}\right)\right\} \\ &\quad + \sum_{i \in C} \log\left[1 - \exp\left\{-\gamma \exp\left[-\left(\frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}}{\sigma}\right)\right]\right\}\right], \end{aligned} \tag{7}$$

where r is the observed number of failures. The maximum likelihood estimates (MLEs), say $\widehat{\boldsymbol{\theta}}$, of the model parameters in $\boldsymbol{\theta} = (\gamma, \sigma, \boldsymbol{\beta}^T)^T$ can be obtained by maximizing the log-likelihood function (7). After fitting model (6), the survival function for y_i can be estimated by

$$S(y_i; \hat{\gamma}, \hat{\sigma}, \widehat{\boldsymbol{\beta}}^T) = 1 - \exp\left\{-\hat{\gamma} \exp\left[-\left(\frac{y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}}{\hat{\sigma}}\right)\right]\right\}.$$

Under conditions that are fulfilled for the parameter vector $\boldsymbol{\theta}$ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is multivariate normal $N_{p+2}(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\widehat{\boldsymbol{\theta}}$ can be approximated by the inverse of the $(p+1) \times (p+1)$ observed information matrix $-\ddot{\mathbf{L}}_{\theta\theta}$ and then the asymptotic inference for the parameter vector $\boldsymbol{\theta}$ can be based on the normal approximation

$N_{p+2}(0, -\ddot{\mathbf{L}}_{\theta\theta}^{-1})$ for $\widehat{\boldsymbol{\theta}}$. The elements of the observed information matrix

$$-\ddot{\mathbf{L}}_{\theta\theta} = \{-\ddot{L}_{r,s}\} = \begin{pmatrix} -\mathbf{L}_{\gamma\gamma} & -\mathbf{L}_{\gamma\sigma} & -\mathbf{L}_{\gamma\beta_j} \\ . & -\mathbf{L}_{\sigma\sigma} & -\mathbf{L}_{\sigma\beta_j} \\ . & . & -\mathbf{L}_{\beta_j\beta_s} \end{pmatrix}$$

where the submatrices in $-\ddot{L}_{\theta\theta}$.

The Hessian matrix for LGIW regression models $\ddot{\mathbf{L}}(\boldsymbol{\beta})$ is made as folow.
We derive the necessary formulas to obtain the second-order partial derivatives of the log-likelihood function. After some algebraic manipulations, we obtain

$$\mathbf{L}_{\gamma\gamma} = -\frac{r}{\sigma^2} - \sum_{i\in C} \exp(-2z_i) h_i \big[1-h_i\big]^{-2};$$

$$\begin{aligned}\mathbf{L}_{\gamma\sigma} &= \sum_{i\in F} \exp(-z_i)(\dot{z}_i)_\sigma + \sum_{i\in C} h_i \exp(-z_i)(\dot{z}_i)_\sigma[\gamma\exp(-z_i)][1-h_i]^{-1} \\ &+ \sum_{i\in C} h_i^2 \gamma \exp(-2z_i)(\dot{z}_i)_\sigma[1-h_i^{-2}];\end{aligned}$$

$$\begin{aligned}\mathbf{L}_{\gamma\beta_j} &= \sum_{i\in F} \exp(-z_i)(\dot{z}_i)_{\beta_j} + \sum_{i\in C} h_i \exp(-z_i)(\dot{z}_i)_{\beta_j}[\gamma\exp(-z_i)][1-h_i]^{-1} \\ &+ \sum_{i\in C} h_i^2 \gamma \exp(-2z_i)(\dot{z}_i)_{\beta_j}[1-h_i^{-2}];\end{aligned}$$

$$\begin{aligned}\mathbf{L}_{\sigma\sigma} &= \frac{r}{\sigma^2} + \sum_{i\in F}\Big\{-(\ddot{z}_i)_{\sigma\sigma} + \gamma\exp(-z_i)\Big[-\big[(\dot{z}_i)_\sigma\big]^2 + (\ddot{z}_i)_{\sigma\sigma}\Big]\Big\} \\ &-\gamma\sum_{i\in C} h_i\exp(-z_i)\Big\{\big[(\dot{z}_i)_\sigma\big]^2[\gamma\exp(-z_i)-1] + (\ddot{z}_i)_{\sigma\sigma}\Big\}[1-h_i]^{-1} \\ &-\gamma\sum_{i\in C} h_i^2\exp(-2z_i)\big[(\dot{z}_i)_\sigma\big]^2[1-h_i]^{-2};\end{aligned}$$

$$\begin{aligned}\mathbf{L}_{\sigma\beta_j} &= -\sum_{i\in F}(\ddot{z}_i)_{\beta_j\sigma} + \gamma\sum_{i\in F}\Big\{-\exp(-z_i)(\dot{z}_i)_{\beta_j}(\dot{z}_i)_\sigma + \exp(-z_i)(\ddot{z}_i)_{\beta_j\sigma}\Big\} \\ &-\gamma\sum_{i\in C} h_i\exp(-z_i)\Big\{(\dot{z}_i)_{\beta_j}(\dot{z}_i)_\sigma[\gamma\exp(-z_i)-1] + (\ddot{z}_i)_{\beta_j\sigma}\Big\}[1-h_i]^{-1} \\ &+\gamma\sum_{i\in C} h_i^2\exp(-2z_i)(\dot{z}_i)_{\beta-j}(\dot{z}_i)_\sigma[1-h_i]^{-2};\end{aligned}$$

and

$$
\begin{aligned}
\mathbf{L}_{\beta_j\beta_s} &= \sum_{i\in F}\Big\{-\gamma\exp(-z_i)(\dot{z}_i)_{\beta_j}(\dot{z}_i)_{\beta_s}\Big\}\\
&\quad -\gamma\sum_{i\in C} h_i\exp(-z_i)\Big\{(\dot{z}_i)_{\beta_j}(\dot{z}_i)_{\beta_s}[\gamma\exp(-z_i)-1]\Big\}[1-h_i]^{-1}\\
&\quad -\gamma\sum_{i\in C} h_i^2\exp(-2z_i)(\dot{z}_i)_{\beta_j}(\dot{z}_i)_{\beta_s}[1-h_i]^{-2};
\end{aligned}
$$

where $h_i = \exp\{-\gamma\exp(-z_i)\}$, $(\dot{z}_i)_\sigma = -z_i/\sigma$, $(\dot{z}_i)_{\beta_j} = -x_{ij}/\sigma$, $(\dot{z}_i)_{\beta_s} = -x_{is}/\sigma$, $(\ddot{z}_i)_{\sigma\sigma} = 2z_i/\sigma^2$, $(\ddot{z}_i)_{\beta_j\sigma} = x_{ij}/\sigma^2$ and $z_i = (y_i - \mathbf{x}_i^T\boldsymbol{\beta})/\sigma$.

5 Conclusions

We introduce a three parameter lifetime distribution so-called the generalized inverse Weibull (GIW) distribution which extends several distributions proposed and widely used in the lifetime literature. The model is much more flexible than the inverse Weibull. We also propose a log-generalized inverse Weibull (LGIW) regression model with the presence of censored data as an alternative to model lifetime data when the failure rate function has unimodal shape.

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