Some Corrections of the Score Test Statistic for Gaussian ARMA Models

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Abstract: In this article we compute three corrected score statistic versions: the Bartlett-type correction and the monotone corrected score statistics proposed by Kakizawa (1996) and Cordeiro et al. (1998). These corrected statistics are used to test the null hypothesis concerning some parameter of interest of an ARMA model, assumed to be Gaussian, stationary and invertible. We also consider the situations where nuisance parameters are present. The formulas are written in matrix form, appropriate for the use of symbolic or numerical languages. Some simulation results are also presented for the AR(1), MA(1) and ARMA(1,1) models.
Key words: ARMA models; Bartlett-type correction; chi-square distribution; score statistics; monotone correction; time series.

1 Bartlett-type Factor in Matrix Form

Consider a time series $\{Y_t, t = 0, \pm 1, \ldots\}$ following a model of the ARMA (autoregressive moving average) family, namely,

$$(1 - \phi_1 B - \cdots - \phi_p B^p)Y_t = (1 - \theta_1 B - \cdots - \theta_q B^q)a_t, \tag{1}$$

where $B^k Y_t = Y_{t-k}$ and the a_t's are elements of a sequence of Gaussian non-correlated random variables, with zero mean and variance $\sigma^2 > 0$. Moreover the polynomials in $z \in \mathbb{C}$

$$\begin{aligned} \phi(z) = &\quad 1 - \phi_1 z - \cdots - \phi_p z^p, \\ \theta(z) = &\quad 1 - \theta_1 z - \cdots - \theta_q z^q \end{aligned}$$

have their (non-common) roots outside the unit circle ($|z| > 1$). This means that we have a stationary and invertible model (see Box et al., 1994 for details). The general problem is to test hypotheses about the parameters of interest, for example

$$H : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)} \qquad vs \qquad A : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}, \tag{2}$$

where $\boldsymbol{\beta}_1 = (\beta_1, \ldots, \beta_s)^\top$ is an s-dimensional vector of parameters of interest and the parameters $\boldsymbol{\beta}_2 = (\beta_{s+1}, \ldots, \beta_r)^\top$ are considered fixed or nuisance and $\boldsymbol{\beta}_1^{(0)}$ is a specified vector. Here the vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r)^\top$ include all the parameters ϕ_i's, θ_i's and σ^2.

The total log-likelihood $l(\beta)$ is given by

$$l(\boldsymbol{\beta}) = -\frac{1}{2}\{n \log(2\pi) + \log(|\boldsymbol{\Sigma}|) + \mathbf{Y}^\top \boldsymbol{\Sigma}^{-1} \mathbf{Y}\}, \tag{3}$$

where $\mathbf{Y} = (Y_1, \ldots, Y_n)^\top$ and $\boldsymbol{\Sigma}$ depends on the vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)^\top$. The partition of the vector of parameters induces the same partition in the score vector $U(\beta)$, the Fisher information matrix, $\mathbf{K} = K(\beta)$, its inverse $\mathbf{K}^{-1}$, and the matrices $\mathbf{A}$ and $\mathbf{M}$ defined below. So we have $U(\boldsymbol{\beta}) = (U_1^\top(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2), U_2^\top(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2))^\top$,

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix}, \quad \mathbf{K}^{-1} = \begin{pmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & K_{22}^{-1} \end{pmatrix} \tag{4}$$

with $\mathbf{M} = \mathbf{K}^{-1} - \mathbf{A}$, and K_{22}^{-1} is the asymptotic covariance matrix of $\tilde{\boldsymbol{\beta}}_2$ and we suppose that the matrix $\mathbf{K}$ is positive definite. It follows that the score statistics S is defined by

$$S = \tilde{U}_1^\top \tilde{\mathbf{K}}^{11} \tilde{U}_1, \tag{5}$$

where $\mathbf{K}^{11}$ is the asymptotic covariance matrix of $\hat{\boldsymbol{\beta}}_1$, obtained from (4), and the notation $\sim$ is used to indicate functions evaluated at the maximum likelihood estimates, under the null hypothesis, in this case, $\tilde{\boldsymbol{\beta}} = (\boldsymbol{\beta}_1^{(0)\top}, \tilde{\boldsymbol{\beta}}_2^\top)^\top$. For large samples and regularity conditions being satisfied, S has asymptotically a χ^2_s distribution under H.

The expressions that determine the coefficients involved in the Edgeworth expansion of the null distribution of S were given by Harris (1985) and that define the corrected statistic S_B proposed by Cordeiro and Ferrari (1991). In this way the corrected score statistics is determined by

$$S_B = S\{1 - (c + bS + aS^2)\}, \tag{6}$$

where the factor that multiplies S is a Bartlett-type correction, as a function of the statistic itself and of the coefficients a, b, and c, which are of order n^{-1} and given by

$$a = \frac{A_3}{12s(s+2)(s+4)}, \quad b = \frac{A_2 - 2A_3}{12s(s+2)}, \quad c = \frac{A_1 - A_2 + A_3}{12s}, \tag{7}$$

where A_1, A_2 and A_3 are quatities that depend on the cumulants. The corrected statistic S_B has the property that it is distributed according to a χ^2_s when terms of order less than n^{-1} are neglected. In the case that the computation of the $A^{'}$s involve unknown parameters, estimates are given by $\tilde{A} = A(\tilde{\beta})$, $\tilde{\beta}$ being a consistent estimator for β, under H (for example the maximum likelihood estimator) and this does not affect the order of approximation, $O(n^{-3/2})$.

However, the improved statistic S_B is not always a monotone transformation. To overcome this, Kakizawa (1996) suggested a monotone transformation involving the score statistic and the coefficient a, b and c, given by

$$S_K = S_B + \frac{1}{4}\left\{c^2 S + 2bcS^2 + \left(2ac + \frac{4}{3}b^2\right)S^3 + 3abS^4 + \frac{9}{5}a^2 S^5\right\}. \tag{8}$$

Cordeiro et al. (1998) proposed an alternative expression for the improved score statistic which is a monotone transformation of S. It is expressed in terms of the normal distribution function $\Phi(\cdot)$, given by

$$S_C = \sqrt{\frac{\pi}{3a}} \exp\left(\frac{b^2}{3a} - c\right)\left\{\Phi\left(\sqrt{6a}S + \sqrt{\frac{2}{3a}}b\right) - \Phi\left(\sqrt{\frac{2}{3a}}b\right)\right\} \tag{9}$$

if $a > 0$ and by

$$S_C = \frac{1}{2b}\exp(-c)\{1 - \exp(-2bS)\} \tag{10}$$

if $a = 0$ and $b \neq 0$ (a is always non negative). If $a = b = 0$, S_B is a monotone transformation of S and $S_C = S_B$.

The model $ARMA(p,q)$ considered in (1) has a spectral density $f_{\boldsymbol{\beta}}(\lambda)$ given by

$$f_{\boldsymbol{\beta}}(\lambda) = \frac{\sigma^2}{2\pi}\frac{|1 - \sum_{j=1}^q \theta_j e^{ij\lambda}|^2}{|1 - \sum_{j=1}^p \phi_j e^{ij\lambda}|^2} \qquad -\pi < \lambda < \pi,$$

see Brockwell and Davis (1991). Consider the following functions, needed for computing the cumulants:

$$f_{\boldsymbol{\beta}}^{i_1 \ldots i_d}(\lambda) = \frac{\partial^d f_{\boldsymbol{\beta}}(\lambda)}{\partial\beta_{i_1}\ldots\partial\beta_{i_d}}, \quad h_{i_1\ldots i_d}(\lambda) = f_{\boldsymbol{\beta}}^{i_1\ldots i_d}(\lambda) f_{\boldsymbol{\beta}}^{-1}(\lambda),$$

$$\mathcal{U}_{i_1\dots i_d}(\lambda) = \frac{\partial^d log f_{\boldsymbol{\beta}}(\lambda)}{\partial\beta_{i_1}\dots\partial\beta_{i_d}}, i_j \in \{1,\dots,r\} \text{ and } d \in \{1,\dots,4\},$$

$$h_{i,j}(\lambda) = h_i(\lambda)h_j(\lambda), \quad h_{i,jk}(\lambda) = h_i(\lambda)h_{jk}(\lambda), \quad \text{etc},$$

$$\mathcal{U}_{i,j}(\lambda) = \mathcal{U}_i(\lambda)\mathcal{U}_j(\lambda), \quad \mathcal{U}_{i,jk}(\lambda) = \mathcal{U}_i(\lambda)\mathcal{U}_{jk}(\lambda), \quad \text{etc.}$$

Consider also the integrals of the functions $\mathcal{U}(\lambda)$'s and $h(\lambda)$'s given by

$$\mathbf{I}_{\cdot} = \int_{-\pi}^{\pi} \mathcal{U}_{\cdot}(\lambda)d\lambda = \mathbf{I}(\mathcal{U}_{\cdot}), \quad \mathcal{I}_{\cdot} = \int_{-\pi}^{\pi} h_{\cdot}(\lambda)d\lambda = \mathcal{I}(h_{\cdot}).$$

To operate with the integrals $\mathbf{I}$ and $\mathcal{I}$ as functions of $\mathcal{U}$'s and h's, respectively, we shall use the notation

$$\mathbf{I}_{i,j} = \mathbf{I}(\mathcal{U}_{i,j}), \quad \mathbf{I}_{ij} = \mathbf{I}(\mathcal{U}_{ij}), \quad \mathbf{I}_{i,j,k} = \mathbf{I}(\mathcal{U}_{i,j,k}), \quad \mathbf{I}_{i,jk} = \mathbf{I}(\mathcal{U}_{i,jk}), \quad \mathbf{I}_{ijk} = \mathbf{I}(\mathcal{U}_{ijk}), \quad \text{etc},$$

$$\mathcal{I}_{i,j} = \mathcal{I}(h_{i,j}), \quad \mathcal{I}_{ij} = \mathcal{I}(h_{ij}), \quad \mathcal{I}_{i,j,k} = \mathcal{I}(h_{i,j,k}), \quad \mathcal{I}_{i,jk} = \mathcal{I}(h_{i,jk}), \quad \mathcal{I}_{i,j,k} = \mathcal{I}(h_{i,j,k}), \quad \text{etc.}$$

The solution of the integrals $\mathbf{I}$ and $\mathcal{I}$ is carried out by tranforming the spectral density of an $ARMA(p,q)$ model, $f_\beta(\lambda)$, as follows:

$$f_\tau(\lambda) = \frac{\sigma^2}{2\pi}\frac{\prod_{j=1}^{p}|1-\delta_j e^{i\lambda}|^2}{\prod_{j=1}^{q}|1-\rho_j e^{i\lambda}|^2}, \qquad -\pi < \lambda < \pi$$

where $\tau = (\rho_1,\dots,\rho_p,\delta_1\dots,\delta_q,\sigma^2)^\top$ and the ρ_i's, δ_i's denote the non-common roots of the characteristic polynomials $\phi(\cdot)$ and $\theta(\cdot)$ respectively, which may be complex. We consider the case of real roots in what follows.

To write the Bartlett-type correction factor in matrix form we proceed as in Cordeiro and Ferrari (1991). First we simplify some terms involved in the coefficients A_1 and A_2, using the cumulants approximated in terms of the integrals $\mathbf{I}$'s provided in Lagos and Morettin (2004). The remaining elements are written using the "exact" cumulants κ's (although in the special models we use the approximations). For $\kappa_{ijk} + 2\kappa_{i,jk}$, $\kappa_{i,jk} - \kappa_{i,j,k}$, $\kappa_{i,j,k,l} - \kappa_{i,j,kl}$, we have

$$\begin{aligned}
\kappa_{ijk} + 2\kappa_{i,jk} &= \delta_{ijk} \approx \tfrac{1}{4\pi}\Big(\mathbf{I}_{i,jk} - \mathbf{I}_{i,j,k} - \mathbf{I}_{j,ik} - \mathbf{I}_{k,ij}\Big),\\
\kappa_{i,jk} - \kappa_{i,j,k} &= \xi_{ijk} \approx \tfrac{1}{4\pi}\Big(\mathbf{I}_{i,jk} - 3\mathbf{I}_{i,j,k}\Big),\\
\kappa_{i,j,k,l} + \kappa_{i,j,kl} &= \omega_{ijkl} \approx \tfrac{1}{2\pi}\Big(2\mathbf{I}_{i,j,k,l} + \mathbf{I}_{i,j,kl}\Big).
\end{aligned} \tag{11}$$

With these coefficients we can write $A_1 = A_{11} + A_{12} + A_{13} + A_{14}$, $A_2 = A_{21} + A_{22} + A_{23} + A_{24}$ and $A_3 = A_{31} + A_{32}$, where

$$\begin{array}{ll}
A_{11} = 3\Sigma'\delta_{ijk}\delta_{kst}a_{ij}a_{st}m_{kr}, & A_{12} = -6\Sigma'\delta_{ijk}\kappa_{r,s,t}a_{ij}a_{kr}m_{st},\\
A_{13} = 6\Sigma'\xi_{ijk}\delta_{rst}a_{js}a_{kt}m_{ir}, & A_{14} = -6\Sigma'\omega_{ijkr}a_{kr}m_{ij},\\
A_{21} = -3\Sigma'\kappa_{i,j,k}\kappa_{r,s,t}a_{kr}m_{ij}m_{st}, & A_{22} = 6\Sigma'\delta_{ijk}\kappa_{r,s,t}a_{ij}m_{kr}m_{st},\\
A_{23} = -6\Sigma'\kappa_{i,j,k}\kappa_{r,s,t}a_{kt}m_{ir}m_{js}, & A_{24} = 3\Sigma'\kappa_{i,j,k,r}m_{ij}m_{kr},\\
A_{31} = 3\Sigma'\kappa_{i,j,k}\kappa_{r,s,t}m_{ij}m_{kr}m_{st}, & A_{32} = 2\Sigma'\kappa_{i,j,k}\kappa_{r,s,t}m_{ir}m_{js}m_{kt}.
\end{array} \tag{12}$$

Now writing the following matrices

$$\mathbf{N}^{(r)} = \Big(\kappa_{r,s,t}\Big), \quad \mathbf{K}_1^{(ij)} = \Big(\kappa_{i,j,k,r}\Big), \quad \boldsymbol{\Delta}^{(k)} = \Big(\delta_{ijk}\Big), \quad \overline{\boldsymbol{\Delta}}^{(i)} = \Big(\delta_{ijk}\Big), \qquad \boldsymbol{\Xi}^{(k)} = \Big(\xi_{ijk}\Big),$$

$$\boldsymbol{\Omega}^{(ij)} = \Big(\omega_{ijkr}\Big), \quad \mathbf{Tr}^{11} = \Big(tr(\mathbf{A}\boldsymbol{\Delta}^{(k)})tr(\mathbf{A}\overline{\boldsymbol{\Delta}}^{(r)})\Big), \quad \mathbf{Tr}^{12} = \Big(tr(\mathbf{A}\boldsymbol{\Delta}^{(k)})tr(\mathbf{M}\mathbf{N}^{(r)})\Big),$$

$$\mathbf{Tr}^{13} = \Big(tr(\mathbf{A}\Xi^{(k)}\mathbf{M}\boldsymbol{\Delta}^{(t)})\Big),\quad \mathbf{Tr}^{14} = \Big(tr(\mathbf{A}\boldsymbol{\Omega}^{(ij)})\Big),\quad \mathbf{Tr}^{21} = \Big(tr(\mathbf{M}\mathbf{N}^{(k)})tr(\mathbf{M}\mathbf{N}^{(r)})\Big),$$

$$\mathbf{Tr}^{22} = \Big(tr(\mathbf{A}\boldsymbol{\Delta}^{(k)})tr(\mathbf{M}\mathbf{N}^{(r)})\Big),\quad \mathbf{Tr}^{23} = \Big(tr(\mathbf{M}\mathbf{N}^{(k)}\mathbf{M}\mathbf{N}^{(t)})\Big),\quad \mathbf{Tr}^{24} = \Big(tr(\mathbf{M}\mathbf{K}_1^{(ij)})\Big),$$

where $tr(\mathbf{A})$ denotes the trace of $\mathbf{A}$ and the matrices $\boldsymbol{\Delta}^{(k)}$ and $\overline{\boldsymbol{\Delta}}^{(i)}$ differ in the fixed index. Then we can write the coefficients A_{ij}'s as follows:

$$A_{11} = 3tr(\mathbf{M}\mathbf{Tr}^{11}),\quad A_{12} = -6tr(\mathbf{A}\mathbf{Tr}^{12}),\quad A_{13} = 6tr(\mathbf{A}\mathbf{Tr}^{13}),\quad A_{14} = -6tr(\mathbf{M}\mathbf{Tr}^{14}),$$

$$A_{21} = -3tr(\mathbf{A}\mathbf{Tr}^{21}),\quad A_{22} = 6tr(\mathbf{M}\mathbf{Tr}^{22}),\quad A_{23} = -6tr(\mathbf{A}\mathbf{Tr}^{23}),$$

$$A_{24} = 3tr(\mathbf{M}\mathbf{Tr}^{24}),\quad A_{31} = 3tr(\mathbf{M}\mathbf{Tr}^{21}),\quad A_{32} = 2tr(\mathbf{M}\mathbf{Tr}^{23}).$$

Finally, for the coefficients A_1, A_2 and A_3, we have

$$\begin{aligned}
A_1 &= 3tr\Big(\mathbf{M}\Big[\mathbf{Tr}^{11} - 2\mathbf{Tr}^{14}\Big]\Big) - 6tr\Big(\mathbf{A}\Big[\mathbf{Tr}^{12} - \mathbf{Tr}^{13}\Big]\Big),\\
A_2 &= -3tr\Big(\mathbf{A}\Big\{\mathbf{Tr}^{21} + 2\mathbf{Tr}^{23}\Big\} - \mathbf{M}\Big\{2\mathbf{Tr}^{22} + \mathbf{Tr}^{24}\Big\}\Big),\\
A_3 &= tr\Big(\mathbf{M}\Big\{3\mathbf{Tr}^{21} + 2\mathbf{Tr}^{23}\Big\}\Big).
\end{aligned} \tag{13}$$

These coefficients are evaluated at the maximum likelihood estimates restricted to the null hypothesis. Then, the coefficients given in (9) are computed in order to obtain the corrected statistic S_B given in (6).

Moreover, we use the rejection rate via simulation to compare the performances of the three corrected score statistics, S_B, S_K, S_C and its original form. We can also compare the power functions of the tests, under a sequence of contiguous alternatives.

2 Applications

In order to illustrate in detail the computation of the Bartlett-type correction factor in matrix form, we consider an $AR(1)$ model, given in (1) with $p = 1$ and $q = 0$. The problem is to test $H_1 : \phi = \phi^{(0)}$ against $A : \phi \neq \phi^{(0)}$, when σ^2 is considered as a nuisance parameter. Note that the vector of parameters in question is $\beta = (\phi, \sigma^2)^\top$, with $|\phi| < 1$, $\beta_1 = \phi$ and $\beta_2 = \sigma^2$. To obtain the matrices in this special case the software `MAPLE V` was used. The matrices reduce to

$$\mathbf{K} = \begin{pmatrix} \frac{1}{1-\phi^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix},\ \mathbf{K}^{-1} = \begin{pmatrix} 1-\phi^2 & 0 \\ 0 & 2\,\sigma^4 \end{pmatrix},\ \mathbf{M} = \begin{pmatrix} 1-\phi^2 & 0 \\ 0 & 0 \end{pmatrix},\ \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 2\,\sigma^4 \end{pmatrix},$$

$$\boldsymbol{\Delta}^{(1)} = \begin{pmatrix} -\frac{4\phi}{(1-\phi^2)^2} & -\frac{1}{\sigma^2(1-\phi^2)} \\ -\frac{1}{\sigma^2(1-\phi^2)} & 0 \end{pmatrix},\quad \boldsymbol{\Delta}^{(2)} = \begin{pmatrix} -\frac{1}{\sigma^2(1-\phi^2)} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\Xi}^{(1)} = \begin{pmatrix} -\frac{8\phi}{(1-\phi^2)^2} & -\frac{3}{\sigma^2(1-\phi^2)} \\ -\frac{3}{\sigma^2(1-\phi^2)} & 0 \end{pmatrix},\quad \boldsymbol{\Xi}^{(2)} = \begin{pmatrix} -\frac{3}{\sigma^2(1-\phi^2)} & 0 \\ 0 & -\frac{2}{\sigma^6} \end{pmatrix},$$

$$\boldsymbol{\Omega}^{(11)} = \begin{pmatrix} \frac{2(7+17\phi^2)}{(1-\phi^2)^3} & \frac{12\phi}{\sigma^2(1-\phi^2)^2} \\ \frac{12\phi}{\sigma^2(1-\phi^2)^2} & \frac{2}{\sigma^4(1-\phi^2)} \end{pmatrix},\quad \boldsymbol{\Omega}^{(12)} = \begin{pmatrix} \frac{14\phi}{\sigma^2(1-\phi^2)^2} & \frac{4}{\sigma^2(1-\phi^2)} \\ \frac{4}{\sigma^2(1-\phi^2)} & 0 \end{pmatrix} = \boldsymbol{\Omega}^{(21)},$$

$$\mathbf{\Omega}^{(22)} = \begin{pmatrix} \frac{4}{\sigma^2(1-\phi^2)} & 0 \\ 0 & \frac{1}{\sigma^8} \end{pmatrix}, \quad \mathbf{N}^{(1)} = \begin{pmatrix} \frac{6\phi}{(1-\phi^2)^2} & \frac{2}{\sigma^2(1-\phi^2)} \\ \frac{2}{\sigma^2(1-\phi^2)} & 0 \end{pmatrix}, \quad \mathbf{N}^{(2)} = \begin{pmatrix} \frac{2}{\sigma^2(1-\phi^2)} & 0 \\ 0 & \frac{1}{\sigma^6} \end{pmatrix}.$$

With these we get

$$\mathbf{Tr}^{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{Tr}^{12} = \mathbf{Tr}^{22}, \quad \mathbf{Tr}^{13} = \begin{pmatrix} \frac{6}{1-\phi^2} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Tr}^{14} = \begin{pmatrix} \frac{4}{1-\phi^2} & 0 \\ 0 & \frac{2}{\sigma^4} \end{pmatrix}$$

$$\mathbf{Tr}^{21} = \begin{pmatrix} \frac{36\phi^2}{(1-\phi^2)^2} & \frac{12\phi}{(1-\phi^2)\,\sigma^2} \\ \frac{12\phi}{(1-\phi^2)\,\sigma^2} & \frac{4}{\sigma^4} \end{pmatrix} = \mathbf{Tr}^{23} \text{ and } \mathbf{Tr}^{24} = \begin{pmatrix} \frac{6(3+7\,\phi^2)}{(1-\phi^2)^2} & \frac{18\phi}{(1-\phi^2)\,\sigma^2} \\ \frac{18\phi}{(1-\phi^2)\,\sigma^2} & \frac{6}{\sigma^4} \end{pmatrix}.$$

From these computed matrices we obtain

$$S_B = S\left\{1 - \frac{1}{n}\left(-\frac{1}{2} - \frac{1+9\phi^2}{2(1-\phi^2)}S + \frac{\phi^2}{1-\phi^2}S^2\right)\right\},$$

from which we notice that the correction does not depend of the nuisance parameter σ^2, evaluating S_B, S_K and S_C for $\phi = \phi^{(0)}$.

A summary of the cases considered, for which the correction factor was calculated is given in Table 1. The results for a moving average model of order 1, $MA(1)$, are the same as for the $AR(1)$ case. The elements of the matrices become complicated as the number of parameters increases. One example is the case of an $ARMA(1,1)$ model, where we want to test the hypothesis $H : \phi = \phi^{(0)}$ against $A : \phi \neq \phi^{(0)}$, the vector of nuisance parameters being $(\theta, \sigma^2)^\top$. We get for the coefficients A's, a, b and c:

$$A_1 = \frac{24(\phi^2\theta^2-2+2\phi\theta-\phi^2)}{n(\phi\theta-1)^2},\ A_2 = -\frac{18(1+\phi^2-2\phi\theta+22\phi^3\theta-11\phi^2\theta^2+\phi^4\theta^2-12\phi^4)}{n(1-\phi^2)(\phi\theta-1)^2},\ A_3 = \frac{180\phi^2(-2\phi\theta+2\phi^2+\theta^2)}{n(1-\phi^2)(\phi\theta-1)^2},$$
$$a = \frac{\phi^2(-2\phi\theta+\phi^2+\theta^2)}{n(1-\phi^2)(\phi\theta-1)^2},\ b = -\frac{1+\phi^2-2\phi\theta-18\phi^3\theta+9\phi^2\theta^2+\phi^4\theta^2+8\phi^4}{2n(1-\phi^2)(\phi\theta-1)^2},\ c = \frac{\phi^2\theta^2+2\phi^2+2\phi\theta-5}{2n(\phi\theta-1)^2}.$$

We observe that this correction also does not depend of the nuisance parameter σ^2, a situation that was also noticed for the corrections for the likelihood ratio statistic (Lagos and Morettin, 2004).

Table 1: Bartlett-type correction factor.

Model	Parameter		A's			coefficients of the polynomial in S		
	interest	nuisance	A_1	A_2	A_3	a	b	c
$AR(1)$	ϕ or θ	none	0	$\frac{12\phi}{n}$	$\frac{180\phi^2}{n(1-\phi^2)}$	$\frac{\phi^2}{n(1-\phi^2)}$	$-\frac{\phi(\phi^2+30\phi-1)}{3n(1-\phi^2)}$	$\frac{\phi(\phi^2+15\phi-1)}{n(1-\phi^2)}$
or	σ^2	none	0	$-\frac{36\sigma^2}{n}$	$\frac{24\sigma^2}{n}$	$\frac{2\sigma^2}{15n}$	$-\frac{7\sigma^2}{3n}$	$\frac{5\sigma^2}{n}$
$MA(1)$	ϕ or θ	σ^2	$-\frac{24}{n}$	$\frac{18(11\phi^2-1)}{n(1-\phi^2)}$	$\frac{180\phi^2}{n(1-\phi^2)}$	$\frac{\phi^2}{n(1-\phi^2)}$	$-\frac{1+9\phi^2}{2n(1-\phi^2)}$	$-\frac{1}{2n}$
	σ^2	ϕ	$-\frac{6}{n}$	$\frac{12}{n}$	$\frac{40}{n}$	$\frac{2}{9n}$	$-\frac{17}{9n}$	$\frac{11}{6n}$

A simulation study to investigate the performance of the corrected score statistics, S_B, S_K and S_C is presented. This performance is measured by the estimated rejection rate and the power of the test for the autoregressive model of order one, moving average model of order one and autoregressive moving average model of order (1,1). The results are showed in eight tables. Table 2 illustrates the simulations.

3 Conclusions

We can observe from the simulations that the results are coherent with what is expected, concerning the fixed parameters under H, the length of the series and the adopted significance levels.

For the estimated rejection rate, we observe that is clear the discrimination of results concerning stationarity and non-stationarity, invertibility and non-invertibility, small and large values of n, nominal levels smaller and bigger than 5%.

The corrected statistics have a better performance than the original statistic S. We also notice the following behavior about the rejection rate ($\tilde{p}$) of the corrected statistics: $\widetilde{p}_{S_B} < \widetilde{p}_{S_K} < \widetilde{p}_{S_C}$.

Concerning the estimated power, $\widetilde{\pi}$, it is possible to conclude more clearly the good performance of the corrected statistics compared with the original statistic, with the remark that if $n \to \infty$, $\theta \to 1$ and $\alpha \to 0$, S has a larger power than S_B and if $|\theta| = 0.6$ and $\alpha = 1\%$ the corrections do not have effect on the power, comparing to S. Lastly it can be observed that in general $\widetilde{\pi}_S < \widetilde{\pi}_{S_B} < \widetilde{\pi}_{S_K} < \widetilde{\pi}_{S_C}$.

Table 2: Estimated rejection rates for S, S_B, S_K, S_C, AR(1) model, ϕ is the interest, $\sigma^2 = 1$ is the nuisance

		$n = 20$				$n = 30$				$n = 40$				$n = 50$			
$\phi^{(0)}$	$\alpha \to$	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%	10%	5%	2.5%	1%
	S	6.44	4.70	3.46	2.39	6.61	4.82	3.63	2.27	6.94	4.77	3.45	2.30	6.85	4.67	3.34	2.21
-0.9	S_B	8.79	6.25	4.62	2.67	8.44	5.48	3.43	0.00	8.41	5.17	3.07	0.00	8.24	4.66	2.52	0.00
	S_K	14.02	10.30	9.07	7.55	12.58	9.38	7.61	5.87	11.96	8.45	6.94	3.11	11.40	7.76	6.11	2.45
	S_C	18.45	13.05	10.72	9.58	15.19	10.60	8.95	7.21	13.57	9.34	7.71	5.51	12.51	8.39	6.62	0.00
	S	6.57	3.74	2.25	1.25	7.54	4.04	2.41	1.33	7.89	3.82	2.27	1.20	8.65	4.17	2.50	1.40
-0.6	S_B	9.25	4.73	2.42	0.00	9.56	4.54	2.50	0.00	9.65	4.46	2.15	0.00	10.01	5.00	2.41	0.00
	S_K	10.05	5.64	3.51	0.61	10.20	5.16	3.27	0.51	10.09	4.93	2.68	0.98	10.30	5.39	2.87	1.15
	S_C	10.27	5.81	3.66	1.21	10.31	5.24	3.31	1.16	10.15	5.06	2.70	1.06	10.36	5.46	2.87	1.23
	S	8.14	3.58	1.76	0.76	8.71	3.76	1.82	0.74	9.15	4.44	2.02	0.84	8.63	4.33	2.09	0.80
-0.3	S_B	9.90	5.10	2.48	1.19	9.97	4.69	2.41	1.00	10.09	5.23	2.58	1.06	9.40	4.78	2.41	0.95
	S_K	10.00	5.12	2.55	1.19	9.99	4.71	2.43	1.00	10.11	5.24	2.60	1.06	9.41	4.78	2.41	0.95
	S_C	10.08	5.17	2.61	1.21	10.05	4.76	2.47	1.01	10.11	5.24	2.62	1.06	9.42	4.81	2.41	0.95
	S	8.22	3.56	1.39	0.39	8.93	4.19	2.07	0.81	9.13	4.34	1.90	0.70	9.28	4.39	2.02	0.74
0.0	S_B	9.65	4.81	2.28	0.88	10.04	5.02	2.69	1.16	9.90	5.09	2.57	0.96	9.86	4.88	2.51	0.97
	S_K	9.70	4.84	2.31	0.92	10.04	5.04	2.70	1.18	9.91	5.10	2.58	0.97	9.87	4.89	2.51	0.97
	S_C	9.76	4.88	2.36	0.95	10.05	5.07	2.73	1.22	9.92	5.11	2.59	0.99	9.88	4.90	2.53	0.98
	S	8.32	3.72	1.70	0.57	8.91	4.00	1.95	0.82	9.08	4.50	2.08	0.79	9.23	4.31	1.98	0.76
0.3	S_B	10.29	5.14	2.54	1.04	10.29	5.20	2.56	1.14	10.17	5.28	2.54	1.01	9.92	4.99	2.34	0.85
	S_K	10.35	5.16	2.57	1.05	10.30	5.23	2.58	1.15	10.19	5.32	2.55	1.02	9.93	4.99	2.35	0.86
	S_C	10.46	5.25	2.59	1.08	10.36	5.25	2.60	1.18	10.21	5.33	2.55	1.03	9.93	4.99	2.35	0.86
	S	7.01	3.65	2.29	1.28	7.77	3.99	2.28	1.27	7.93	3.84	2.11	1.11	8.62	4.24	2.45	1.26
0.6	S_B	9.85	5.04	2.46	0.00	9.55	4.91	2.40	0.00	9.72	4.66	2.11	0.00	10.31	4.85	2.38	0.00
	S_K	10.70	5.95	3.40	0.61	10.01	5.44	3.09	0.41	10.08	5.03	2.61	0.81	10.60	5.16	2.77	1.04
	S_C	10.92	6.17	3.54	1.21	10.18	5.48	3.12	1.09	10.18	5.15	2.69	1.01	10.63	5.16	2.77	1.13
	S	6.61	4.75	3.53	2.43	6.48	4.63	3.36	2.23	6.62	4.82	3.42	2.38	6.98	4.70	3.53	2.50
0.9	S_B	9.14	6.52	4.82	2.81	8.38	5.51	3.56	0.00	8.11	4.73	2.80	0.00	8.44	4.81	2.30	0.00
	S_K	14.43	10.69	9.34	7.74	12.31	9.18	7.55	5.79	11.74	8.16	6.57	3.13	11.88	8.16	6.10	2.63
	S_C	18.43	13.33	11.07	9.79	14.41	10.36	8.75	7.20	12.99	9.02	7.35	5.45	13.10	8.89	6.69	0.00

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